1. (a) Yes. An $n \times n$ matrix of rank $n$ is invertible, and thus $v = A^{-1}b$ is the solution we seek. Alternatively, you could say that since $A$ is invertible, $Av = b$ has a unique solution $v \in \mathbb{R}^n$.

(b) True. The reason is that 
\[(A - I_n)(A + I_n) = A^2 - I_n A + AI_n - I_n^2 = A^2 - A + A - I_n = A^2 - I_n = -I_n.\]

So then $(A - I_n)(-A - I_n) = I_n$, and similarly $(-A - I_n)(A - I_n) = I_n$. This means that there is a matrix $B$ such that $AB = I_n$ and $BA = I_n$, and so $A - I_n$ is invertible.

(c) False. If $A$ is the matrix of a reflection about a line in $\mathbb{R}^2$, then $A \cdot A$ fixes every $\vec{x} \in \mathbb{R}^2$ (since reflecting twice returns the original vector). Thus $A^2 = I_2$. So in particular the matrix
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
satisfies $A^2 = I_2$ but $A \neq I_2$ and $A \neq -I_2$.

(d) False. For instance, $T$ could be the zero-map, and then $T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = \vec{0}$, and so certainly $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is not linearly independent. For a less extreme example, $T$ could be a linear transformation that maps $\vec{v}_1, \vec{v}_2, \vec{v}_3$ onto the same line.

2. Harry’s numbers are precisely the values $x_1, x_2, x_3, x_4$ that satisfy the following system of equations:

\[
\begin{align*}
x_1 + 2x_2 + x_4 &= 0 \\
-3x_1 + x_3 &= 1 \\
-x_1 + 4x_2 + x_3 + 2x_4 &= 1
\end{align*}
\]

To find the solutions, we need to write the augmented matrix for this system and put it into reduced row-echelon form. The augmented matrix is
\[
\begin{bmatrix}
1 & 2 & 0 & 1 | 0 \\
-3 & 0 & 1 & 0 | 1 \\
-1 & 4 & 1 & 2 | 1
\end{bmatrix}
\]

Adding 3 times the first row to the second row and 1 times the first row to the third row gives
\[
\begin{bmatrix}
1 & 2 & 0 & 1 | 0 \\
0 & 6 & 1 & 3 | 1 \\
0 & 6 & 1 & 3 | 1
\end{bmatrix}
\]

Adding -1 times the second row to the third row and -1/3 times the second row to the first row and then multiplying the second row by 1/6 gives
\[
\begin{bmatrix}
1 & 0 & -1/3 & 0 | -1/3 \\
0 & 1 & 1/6 & 1/2 | 1/6 \\
0 & 0 & 0 & 0 | 0
\end{bmatrix}
\]
This is now in rref. We see that the leading variables are \( x_1 \) and \( x_2 \), while the free variables are \( x_3 \) and \( x_4 \). Setting \( x_3 = t_1 \) and \( x_4 = t_2 \), we get that the set of solutions is

\[
\left\{ \left( \frac{1}{3}t_1 - \frac{1}{3}t_2, \frac{1}{6}t_1 - \frac{1}{2}t_2 + \frac{1}{6}t_1, t_1, t_2 \right) \in \mathbb{R}^4 \mid t_1, t_2 \in \mathbb{R} \right\}
\]

(I used row vector form here instead of column vector form, but the content is the same.) So for every choice of real numbers \( t_1, t_2 \), we get a set of four numbers \((x_1, x_2, x_3, x_4)\) that satisfy Harry’s conditions.

3. (a) Observe first that \( W = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 - x_3 = 0, x_4 = 0\} \). So \( W \) is the set of vectors \( x \in \mathbb{R}^4 \) such that \( Ax = 0 \), with \( A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \). So \( W \) is the kernel of \( A \), and thus is a subspace by a theorem from class.

(b) It seems unlikely to be a subspace since the second condition \( x_4 = x_1x_3 \) involves multiplication of variables and so is not linear. Indeed, we can show this is not closed under addition: \((1, 2, 3, 3) \in W \) and \((-1, 2, 1, -1) \in W \) but their sum \((0, 4, 4, 2)\) is not in \( W \) since it fails the condition \( x_4 = x_1x_3 \). You could also proceed by showing that \( W \) is not closed under scalar multiplication: \((1, 2, 3, 3) \in W \), but \( 2(1, 2, 3, 3) = (2, 4, 6, 6) \) is not in \( W \).

4. Find all the vectors in the kernel of each of the following linear transformations, and justify your answers.

(a) The shear \( T(x) = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} x \).

The reduced row echelon form of this matrix is \( I_2 \), and so it is invertible. Thus the only \( x \) with \( T(x) = 0 \) is \( x = A^{-1}0 = 0 \). So the kernel of \( T \) is \( \{0\} \).

(b) Reflection about a plane in \( \mathbb{R}^3 \).

This transformation is again invertible (it is its own inverse, since \( T(T(x)) = x \) for all \( x \)), so it has kernel \( \{0\} \).

(c) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by projection onto the line \( y = x \).

The kernel consists of the vectors that project to \( 0 \), which are precisely those that lie along the line perpendicular to \( y = x \). That line is \( y = -x \), also known as the span of the vector \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

(d) \( T(x) = Ax \), where \( A \) is an \( n \times m \) matrix of rank \( m \).

We seek all solutions to \( Ax = 0 \). So we put the augmented matrix \([A|0]\) into rref, and we find there is a leading 1 in each column (since the rank of the matrix equals the number of columns). Thus there are no free variables, and it follows that \( Ax = 0 \) has a unique solution. Hence the kernel of \( T \) consists only of \( 0 \).

5. First note that since \( W_1 \) and \( W_2 \) are subspaces of \( \mathbb{R}^n \), we have \( \vec{0} \in W_1 \) and \( \vec{0} \in W_2 \), and so \( \vec{0} \in W_1 \cap W_2 \). Let \( \vec{x} \) and \( \vec{y} \) be vectors in \( W_1 \cap W_2 \). By the definition of intersection, this means that \( \vec{x} \) is in both \( W_1 \) and \( W_2 \), and the same for \( \vec{y} \). Since both \( \vec{x} \) and \( \vec{y} \) are in \( W_1 \) and \( W_1 \) is a
subspace of $\mathbb{R}^n$, and subspaces are closed under addition, we have $\vec{x} + \vec{y} \in W_1$. But both $\vec{x}$ and $\vec{y}$ are in $W_2$ also, and $W_2$ is a subspace of $\mathbb{R}^n$, and so $\vec{x} + \vec{y} \in W_2$. Thus $\vec{x} + \vec{y} \in W_1 \cap W_2$. Similarly, let $r \in \mathbb{R}$. Then $\vec{x} \in W_1$ and $W_1$ is a subspace of $\mathbb{R}^n$, so we have $r\vec{x} \in W_1$. But $\vec{x} \in W_2$ and $W_2$ is also a subspace of $\mathbb{R}^n$, and hence $r\vec{x} \in W_2$. Thus $r\vec{x} \in W_1 \cap W_2$. This shows that $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

6. Suppose that we have $a_1x_1 + \cdots + a_nx_n = 0$ with $x_i \in S$. We know that 0 can also be written as the linear combination

$$0x_1 + \cdots + 0x_n.$$

Since 0 $\in \text{Span}(S)$, our hypothesis gives us that there is only one way to write 0 as a linear combination of vectors in $S$. Therefore we must have $a_1 = 0, \ldots, a_n = 0$. Hence $S$ is linearly independent.