Two Heads Are Better Than None

Or, Me and the Fibonaccis

Steve Kennedy
(with help from Matt Stafford)
Carleton College and
MAA Books

Problems are the lifeblood of mathematics.
— David Hilbert

Indiana MAA Spring Section Meeting 2014

The Problem

The Game:

Flip a fair coin repeatedly until two consecutive heads appear, stop.

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The Game:

Flip a fair coin repeatedly until two consecutive heads appear, stop.

The Problem(s):

- What is the probability that the game will *ever* end? (Intuitively, this is big, but consider HTHTHTHT..., or TTTTTT...)
- What is the probability that the game ends after *exactly* n flips?

Counting Arguments

$n = 2$ $\quad \begin{array}{cc} \color{red}{HH} & TH \\ HT & TT \end{array}$

Each event equally likely, so $P_2 = 1/4$.

Counting Arguments

$n = 2$

HH	TH
HT	TT

Each event is equally likely, so $P_2 = 1/4$.

n=3

HHH	THH
HHT	THT
HTH	TTH
HTT	TTT

Each event is equally likely, so $P_3 = 1/8$.

Counting Arguments: Wait just one minute!

$n = 2$	HH	TH
	HT	TT

Each event is equally likely, so $P_2 = 1/4$.

	HHH	THH
	HHT	THT
n=3	HTH	TTH
	HTT	TTT

Each event is equally likely, so $P_3 = 1/8$.

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So, actually, the probability of seeing THH is:

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This had to work out—because the coin doesn't know the rules of the game.

Counting Arguments

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Each event is equally likely, so $P_2 = 1/4$.

n=3

HHH	THH
HHT	THT
HTH	TTH
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Each event is equally likely, so $P_3 = 1/8$.

n=4

HHHH	HTHH	THHH	TTHH
HHHT	HTHT	THHT	TTHT
HHTH	HTTH	THTH	TTTH
HHTT	HTTT	THTT	TTTT

Each event is equally likely, so $P_4 = 2/16$.

Counting Arguments

n=5

HHHHH	HTHHH	THHHH	TTHHH
HHHHT	HTHHT	THHHT	TTHHT
HHHTH	HTHTH	THHTH	TTHTH
HHHTT	HTHTT	THHTT	TTHTT
HHTHH	HTTHH	THTHH	TTTHH
HHTHT	HTTHT	THTHT	TTTHT
HHTTH	HTTTH	THTTH	TTTTH
HHTTT	HTTTT	THTTT	TTTTT

So, $P_5 = 3/32$.

Counting Arguments

n=5

HHHHH	HTHHH	THHHH	TTHHH
HHHHT	HTHHT	THHHT	TTHHT
HHHTH	HTHTH	THHTH	TTHTH
HHHTT	HTHTT	THHTT	TTHTT
HHTHH	HTTHH	THTHH	TTTHH
HHTHT	HTTHT	THTHT	TTTHT
HHTTH	HTTTH	THTTH	TTTTH
HHTTT	HTTTT	THTTT	TTTTT

So, $P_5 = 3/32$.

n=6 HTHTHH, HTTTHH, THTTHH, TTHTHH, TTTTHH

So, $P_6 = 5/64$.

Counting Arguments

n=5

HHHHH	HTHHH	THHHH	TTHHH
HHHHT	HTHHT	THHHT	TTHHT
HHHTH	HTHTH	THHTH	TTHTH
HHHTT	HTHTT	THHTT	TTHTT
HHTHH	HTTHH	THTHH	TTTHH
HHTHT	HTTHT	THTHT	TTTHT
HHTTH	HTTTH	THTTH	TTTTH
HHTTT	HTTTT	THTTT	TTTTT

So, $P_5 = 3/32$.

n=6 HTHTHH, HTTTHH, THTTHH, TTHTHH, TTTTHH

So, $P_6 = 5/64$.

n=7

Counting Arguments

n=5

HHHHH	HTHHH	THHHH	TTHHH
HHHHT	HTHHT	THHHT	TTHHT
HHHTH	HTHTH	THHTH	TTHTH
HHHTT	HTHTT	THHTT	TTHTT
HHTHH	HTTHH	THTHH	TTTHH
HHTHT	HTTHT	THTHT	TTTHT
HHTTH	HTTTH	THTTH	TTTTH
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So, $P_6 = 5/64$.

n=7 Homework

Search for Pattern

$$\frac{1}{4}, \ \frac{1}{8}, \ \frac{2}{16}, \ \frac{3}{32}, \ \frac{5}{64}, \ \ldots$$

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$P_n = \frac{x_n}{2^n},$ What's x_n?

Search for Pattern

$$\frac{1}{4},\ \frac{1}{8},\ \frac{2}{16},\ \frac{3}{32},\ \frac{5}{64},\ \ldots$$

$P_n = \frac{x_n}{2^n}$, What's x_n?

$$x_n = 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

An Unnecessary(?) Aside

The Fibonacci Numbers — 1202

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The Rules for Rabbit Reproduction

1. Gestation period is one month.

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   (And never die.)

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The Rules for Rabbit Reproduction

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2. Rabbits born in male/female pairs.
3. Rabbits reach sexual maturity in one month.
4. Rabbits are always pregnant.
   (And never die.)

Start with one newborn pair, how many pairs will you have n months later?

An Unnecessary(?) Aside

The Fibonacci Numbers — 1202

Month:	0	1	2	3	4	5
Mature Pairs	0					
Immature Pairs	1					
Total Pairs	1					

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Month:	0	1	2	3	4	5
Mature Pairs	0	1	1			
Immature Pairs	1	0	1			
Total Pairs	1	1	2			

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Month:	0	1	2	3	4	5
Mature Pairs	0	1	1	2		
Immature Pairs	1	0	1	1		
Total Pairs	1	1	2	3		

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Thus, $F_{n+1} = F_n + F_{n-1}$.

An Answer

Theorem

The sequence x_n is the Fibonacci sequence.

Example

	T	HT
n=6	T	HT
	T	

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So, $x_6 \leq x_5 + x_4$.

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Conversely	*THTHH	**TTHH
	*TTTHH	

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Theorem

The sequence x_n is the Fibonacci sequence.

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	THTTHH	HTHTHH
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So, $x_6 \leq x_5 + x_4$.

	THTTHH	HTHTHH
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So, $x_6 \geq x_5 + x_4$.

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- What is the probability that the game ends in four or fewer flips?
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- What is the probability that the game ends after two or three flips?
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- What is the probability that the game ends in four or fewer flips?
$$\frac{1}{4} + \frac{1}{8} + \frac{2}{16} = \frac{1}{2}$$
- What is the probability that the game ends in twenty or fewer flips?

More Questions

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$$\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

- What is the probability that the game ends in four or fewer flips?

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- What is the probability that the game ends in twenty or fewer flips?

$$\sum_{n=2}^{20} P_n \approx .983$$

Original Question

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$$\lim_{n\to\infty} \sum_{k=2}^{n} P_n = \sum_{k=0}^{\infty} \frac{F_k}{2^{k+2}}$$

Does this series converge?

Convergence

$$\frac{1}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{5}{64}+\frac{8}{128}+\ldots \quad = \quad S$$

Convergence

$$
\begin{aligned}
\frac{1}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{5}{64}+\frac{8}{128}+\dots &= S \\
\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\dots &= \frac{1}{2}
\end{aligned}
$$

Convergence

$$\begin{aligned}
\frac{1}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{5}{64}+\frac{8}{128}+\dots &= S\\
\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\dots &= \frac{1}{2}\\
\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\dots &= \frac{1}{8}
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Convergence

$$
\begin{array}{rcl}
\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \dots & = & S \\
\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \dots & = & \frac{1}{2} \\
\frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \dots & = & \frac{1}{8} \\
\frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \dots & = & \frac{1}{16}
\end{array}
$$

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\frac{1}{64}+\frac{1}{128}+\dots & = & \frac{1}{32} \\
\frac{1}{128}+\dots & = & \frac{1}{64} \\
\dots & &
\end{array}
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\dots & &
\end{array}
$$

So, $S = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \dots$.

The Sum

Recall,

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We just decided,

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So,

$$S = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \dots\right.$$

$$S = \frac{1}{2} + \frac{1}{2}S$$

The Sum

Recall,

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$$S = \frac{1}{2} + \frac{1}{2}S$$

$$S = 1$$

The Sum

$$\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \ldots = 1$$

$$\sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} = 1$$

The Sum

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The Sum

$$\sum_{n=2}^{\infty} \frac{F_{n-2}}{2^n} = 1!!!$$

The Miracle

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THIS IS A MIRACLE!!!

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Definition

A *miracle* is the simultaneous occurrence of two or more zero-probability events.

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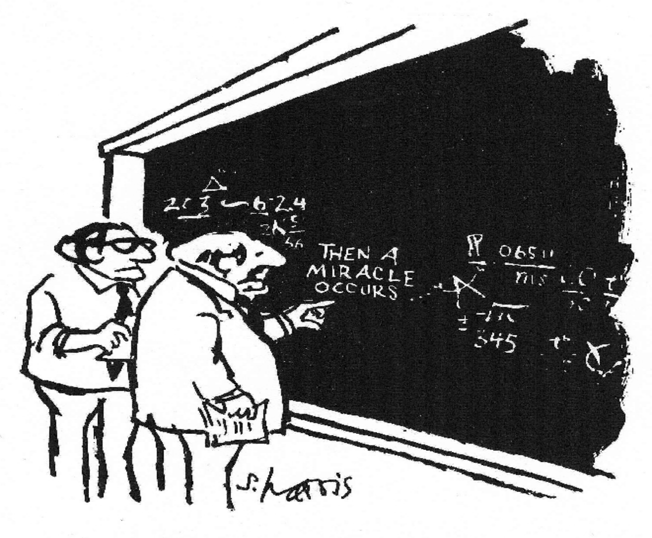
A *miracle* is the simultaneous occurrence of two or more zero-probability events.

- The series converged. (And I could prove it!)
- We could find the value to which it converged.
- That value was rational.
- That value was the simplest possible rational.

The Miracle



"I think you should be more explicit here in step two."

The Very Good Reason

$$g(x) = \sum_{k=0}^{\infty} F_k x^k \quad = \quad 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots$$

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$$x\,g(x) = \sum_{k=0}^{\infty} F_k x^{k+1} \quad = \quad x + x^2 + 2x^3 + 3x^4 + \dots$$

The Very Good Reason

$$
\begin{aligned}
g(x) = \sum_{k=0}^{\infty} F_k x^k &= 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots \\
x\,g(x) = \sum_{k=0}^{\infty} F_k x^{k+1} &= x + x^2 + 2x^3 + 3x^4 + \dots \\
x^2 g(x) = \sum_{k=0}^{\infty} F_k x^{k+2} &= x^2 + x^3 + 2x^4 + \dots
\end{aligned}
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So,

$$g(x) - xg(x) - x^2 g(x) = 1$$

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$$g(x) - xg(x) - x^2 g(x) = 1$$

$$g(x) = \frac{1}{1 - x - x^2}$$

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$$g(x) = \sum_{k=0}^{\infty} F_k x^k = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots$$

$$g\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} F_k \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \dots$$

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$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \dots$$

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$$g\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} F_k \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)^4 + \dots$$

$$= 1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \dots$$

$$= 4\left(\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \dots\right.$$

Technicalities

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(NB The radius of convergence, $\frac{1}{\lambda}$, is approximately .618, so $\frac{1}{2}$ is comfortably inside.)

Two Great Tastes That Taste Great Together

The function, $g(x) = \frac{1}{1-x-x^2}$, is called the generating function for the Fibonacci numbers.

Turning the Page

Rewrite $g(x)$ using partial fractions:

$$\frac{1}{1-x-x^2} = \frac{A}{1-\lambda x} + \frac{B}{1+(\lambda-1)x}$$

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Similarly,

$$\frac{B}{1+(\lambda-1)x} = \sum_{k=0}^{\infty} B(-1)^k\,((\lambda-1)x)^k$$

Never in a Million Years

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Recalling the values of A and B,

$$\sum_{k=0}^{\infty} F_k x^k = \sum_{k=0}^{\infty} \left[\frac{\lambda^{k+1}}{\sqrt{5}} + \frac{(-1)^k(\lambda-1)^{k+1}}{\sqrt{5}}\right] x^k$$

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Power series expansions are unique! So,

$$F_k = \frac{\lambda^{k+1}}{\sqrt{5}} + \frac{(-1)^k(\lambda-1)^{k+1}}{\sqrt{5}}$$

Be wise, generalize!

Heads	Numerators and Recursion	Generating Function
One	$1, 1, 1, 1, \ldots$ $a_n = a_{n-1}$	$\frac{1}{1-x}$
Two	$1, 1, 2, 3, \ldots$ $a_n = a_{n-1} + a_{n-2}$	$\frac{1}{1-x-x^2}$
Three	$1, 1, 2, 4, 7, \ldots$ $a_n = a_{n-1} + a_{n-2} + a_{n-3}$	$\frac{1}{1-x-x^2-x^3}$
Four	$1, 1, 2, 4, 8, 15, \ldots$ $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$	$\frac{1}{1-x-x^2-x^3-x^4}$

And so on ...

"Suppose you had a three-sided coin?"

Heads	Numerators and Recursion	Generating Function
One	$1, 2, 4, 8, \ldots$ $a_n = 2a_{n-1}$	$\frac{1}{1-2x}$
Two	$1, 2, 6, 16, 48, \ldots$ $a_n = 2(a_{n-1} + a_{n-2})$	$\frac{1}{1-2x-2x^2}$
Three	$1, 2, 6, 18, 52, 152, \ldots$ $a_n = 2(a_{n-1} + a_{n-2} + a_{n-3})$	$\frac{1}{1-2x-2x^2-2x^3}$
Four	$1, 2, 6, 18, 54, 160, \ldots$ $a_n = 2(a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4})$	$\frac{1}{1-2x-2x^2-2x^3-2x^4}$

Homework: Do the n consecutive heads from an m-sided coin problem.

Fibonacci Fractions

Do the long division problem:

$$\frac{10,000}{9,899} =$$

Fibonacci Fractions

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What just happened?

$$\begin{aligned} g\left(\frac{1}{100}\right) &= 1 + \frac{1}{100} + \frac{2}{(100)^2} + \frac{3}{(100)^3} + \ldots \\ &= 1 + .01 + .0002 + .000003 + \ldots \end{aligned}$$

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$$g\left(\frac{1}{1000}\right) = \frac{1,000,000}{998,999}$$

Multiplying Weirdness

Recall

$$F_k = \frac{1}{\sqrt{5}}\left[\lambda^{k+1} + (-1)^k(\lambda - 1)^{k+1}\right]$$

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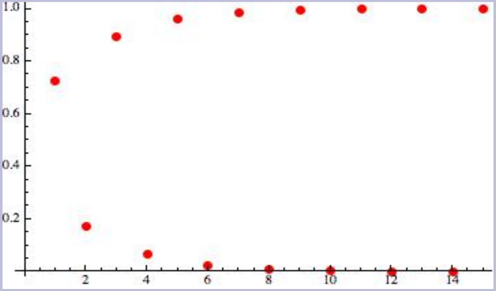
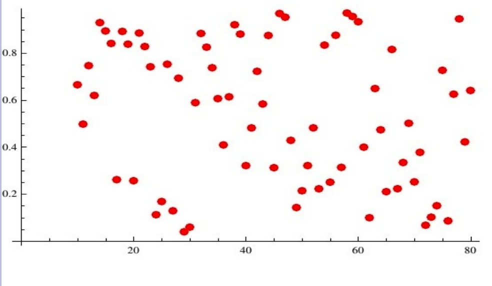
$$F_k = \frac{1}{\sqrt{5}}\left[\lambda^{k+1} + (-1)^k(\lambda - 1)^{k+1}\right]$$

Here's a picture of those fractional parts:



Pictured is the fractional part of $\frac{1}{\sqrt{5}}\lambda^k$, $k = 1, 2, \ldots, 15$.

Your Real Homework



The picture for the fractional part of 1.5^n for $n = 10 \ldots 80$. (This is what we expect to see.)

Your Real Homework

Conjecture

There are not very many real numbers, γ, that have the property that there exists a constant C so that the sequence consisting of the fractional parts of $C\gamma^n$, $n = 1, 2, \ldots$ has only finitely many limit points.

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There are not very many real numbers, γ, that have the property that there exists a constant C so that the sequence consisting of the fractional parts of $C\gamma^n$, $n = 1, 2, \ldots$ has only finitely many limit points.

$$\begin{aligned} \text{Not very many} &= \text{measure zero} \\ &= \text{first category} \\ &= \text{nowhere dense} \end{aligned}$$

Your Real Homework: Hints

The number $1 + \sqrt{2} = p$ is even better than the golden mean.

$$
\begin{aligned}
p^1 &\approx 2.41421 \\
p^2 &\approx 5.82842 \\
p^3 &\approx 14.07106 \\
&\dots \\
p^8 &\approx 1153.99913 \\
p^9 &\approx 2786.00035 \\
p^{10} &\approx 6725.99985
\end{aligned}
$$

Your Real Homework: Hints

$$
\begin{aligned}
p^1 &= 1 + \sqrt{2} \\
p^2 &= 3 + 2\sqrt{2} \\
p^3 &= 7 + 5\sqrt{2} \\
\ldots \\
p^8 &= 577 + 408\sqrt{2} \\
p^9 &= 1393 + 985\sqrt{2} \\
p^{10} &= 3363 + 2378\sqrt{2}
\end{aligned}
$$

A Final Word

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Thank you for your kind attention.