1. Find a basis for the kernel and image of the following matrix:

\[ A = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ -3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 4 & 1 & 2 \end{bmatrix} . \]

To find a basis for the kernel, we need to row reduce the augmented matrix

\[ \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & | & 0 \\ -3 & 0 & 0 & 1 & 0 & | & 0 \\ -1 & 0 & 4 & 1 & 2 & | & 0 \end{bmatrix}, \]

and we get the reduced row-echelon form

\[ \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 1/6 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]

Thus the free variables are \( x_2, x_4, \) and \( x_5, \) and we set \( x_2 = t_1, \) \( x_4 = t_2, \) and \( x_5 = t_3. \) The set of solutions is then

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} t_1 \\ \frac{1}{3}t_2 \\ -\frac{1}{6}t_2 \\ \frac{1}{2}t_3 \\ t_3 \end{bmatrix}. \]

Separating the arbitrary constants \( t_1, t_2, t_3 \) into their own vectors then yields a basis of

\[ \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ -1/6 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}. \]

As for the image, the above reduced row-echelon form shows that the non-redundant columns are the first and the third, so a basis for the image is

\[ \left\{ \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right\}. \]
2. Find a basis for the following subspace of $P_4$.

$$W = \{ p(x) \in P_4 \mid p(1) = p(-1) = 0 \}.$$  

What is the dimension of $W$?

Note that $W$ is the same as

$$\{ a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4 \mid a_4 + a_3 + a_2 + a_1 + a_0 = 0, a_4 - a_3 + a_2 - a_1 + a_0 = 0 \}.$$  

There are different ways to do this problem, but the most systematic is to find all solutions to the system of equations

$$\begin{align*}
a_0 + a_1 + a_2 + a_3 + a_4 &= 0 \\
a_0 - a_1 + a_2 - a_3 + a_4 &= 0.
\end{align*}$$

So we put the augmented matrix

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & | & 0 \\
1 & -1 & 1 & -1 & 1 & | & 0
\end{bmatrix}$$

in reduced row-echelon form. We get

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & | & 0 \\
0 & 1 & 0 & 1 & 0 & | & 0
\end{bmatrix}$$

Thus our free variables are $a_2$, $a_3$, and $a_4$. Set $a_2 = t_1$, $a_3 = t_2$, and $a_4 = t_3$. The set of solutions is then

$$\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix} = \begin{bmatrix}
-t_1 \\
t_1 \\
-t_2 \\
t_2 \\
-t_3
\end{bmatrix}. $$

Separating the arbitrary constants $t_1, t_2, t_3$ into their own vectors then yields a basis of

$$\left\{ \begin{bmatrix}
-1 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix} \right\}.$$  

But we need to re-convert these to polynomials, since $W$ is a subspace of $P_4$, and hence any basis for it should consist of elements of $P_4$. So a basis for $W$ is

$$\{ -1 + x^2, -x + x^3, -1 + x^4 \},$$

and $W$ has dimension 3.
3. Determine whether the following mappings are linear transformations. Either prove that a
given map is linear or give a counterexample to show it’s not linear.

(a) \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2) \)

Not linear since \( 2T((0, 0)) = 2(0, 4, 0) = (0, 8, 0) \) but \( T(2(0, 0)) = T((0, 0)) = (0, 4, 0) \).

(b) \( T : P_2 \to P_3 \) defined by \( T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0 \)

Linear. Let \( \vec{u} = u_2x^2 + u_1x + u_0 \) and \( \vec{v} = v_2x^2 + v_1x + v_0 \). Then

\[
T(\vec{u} + \vec{v}) = (u_0 + v_0)x^3 + (u_1 + v_1 - u_0 - v_0)x^2 + 3(u_2 + v_2) - (1/2)(u_0 + v_0)
\]

A similar calculation shows that \( T(c\vec{u}) = cT(\vec{u}) \).

4. Let \( V \) be a subspace of \( \mathbb{R}^n \) with \( \dim(V) = n \). Explain why \( V = \mathbb{R}^n \).

Since \( \dim(V) = n \), \( V \) has a basis \( B \) containing \( n \) elements. Since \( B \) is a basis for \( V \), \( B \) is linearly independent. However, \( n \) linearly independent vectors in the \( n \)-dimensional space \( \mathbb{R}^n \) must form a basis for \( \mathbb{R}^n \), by Theorem 3.3.4(c). Thus \( B \) is a basis for both \( V \) and \( \mathbb{R}^n \), and so \( \text{Span}(B) = V \) and \( \text{Span}(B) = \mathbb{R}^n \). Hence \( V = \mathbb{R}^n \).

5. (a) Consider the mapping \( T : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2} \) defined by

\[
T\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a+b & c-b \\ b+2d-3c & d+4a \end{array} \right).
\]

Prove that \( T \) is a linear transformation.

Let

\[
\vec{u} = \left( \begin{array}{cc} u_{11} & u_{12} \\ u_{21} & u_{22} \end{array} \right), \quad \vec{v} = \left( \begin{array}{cc} v_{11} & v_{12} \\ v_{21} & v_{22} \end{array} \right)
\]

Then

\[
T(\vec{u} + \vec{v}) = \left( \begin{array}{cc} u_{11} + v_{11} + u_{12} + v_{12} & \left( u_{21} + v_{21} - \left( u_{12} + v_{12} \right) \right) \\ u_{12} + v_{12} + 2(u_{22} + v_{22}) - 3(u_{21} + v_{21}) & u_{22} + v_{22} + 4(u_{11} + v_{11}) \end{array} \right)
\]

A similar calculation shows that \( T(c\vec{u}) = cT(\vec{u}) \).
(b) Given the basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ of $\mathbb{R}^{2 \times 2}$, give the matrix $[T]_\alpha$ of $T$ with respect to the basis $\alpha$.

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the first column of $[T]_\alpha$ is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}.$$  

Proceeding similarly with the other columns, we get

$$[T]_\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

(c) Show that $T$ is an isomorphism, and use the determinant in your solution.

To show $T$ is an isomorphism, one method is to show directly that $T$ is invertible. We can accomplish this by showing that $[T]_\alpha$ from part (b) is an invertible matrix. This, in turn, we can do by finding the determinant of $[T]_\alpha$. Expand along the first column to get

$$\det[T]_\alpha = \det \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix} - 4 \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

$$= \det \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix} - 4(1 \cdot 1 \cdot 2)$$

$$= (3 - 1) - 8 = -6$$

Since this determinant is non-zero, the matrix is invertible, and thus $T$ is an isomorphism.

6. The mapping $T : \mathbb{R}^2 \to P_2$ given by $T \left( \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right) = (a_1 + a_2)x^2 + a_2x + a_1$ is a linear transformation.

(a) Prove that $\alpha = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$ and $\beta = \{x^2 + 2, x^2 + x, 1\}$ is a basis for $P_2$. 

4
First note that $\alpha$ is a basis for $\mathbb{R}^2$ since $\alpha$ is linearly independent (it is a set of two vectors and neither is a multiple of the other) and the dimension of $\mathbb{R}^2$ is two. Similarly, one can check that $\beta$ is linearly independent by assuming that $c_1(x^2 + 2) + c_2(x^2 + x) + c_3(1) = \mathbf{0}$, setting up a system of equations involving $c_1, c_2, c_3$, and showing that $c_1 = c_2 = c_3 = 0$. Since $P_2(\mathbb{R})$ has dimension 3 this shows that $\beta$ is a basis, by Theorem 3.3.4(c).

(b) Find the matrix $[T]^\beta_\alpha$.

To compute the first column of $[T]^\beta_\alpha$, we find $T((1, 2))$ and write it in $\beta$-coordinates. We have

$$T((1, 2)) = 3x^2 + 2x + 1 = 1(x^2 + 2) + 2(x^2 + x) - 1(1)$$

and thus the first column of $[T]^\beta_\alpha$ is

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$  

Note that to find the $\beta$-coordinates of $3x^2 + 2x + 1$, you can either eyeball a solution as follows: only $x^2 + x$ involves $x$, and so we must have $2(x^2 + x)$, and then the only other term with $x^2$ is $x^2 + 2$, so we must add on $1(x^2 + 2)$ to get $3x^2$, then we have a constant term of 2, so we must subtract 1(1). Or you can take the more systematic approach of writing

$$3x^2 + 2x + 1 = a_1(x^2 + 2) + a_2(x^2 + x) + a_3(1),$$

which leads to the system of equations

$$a_2 + a_1 = 3$$
$$a_2 = 2$$
$$a_3 + 2a_1 = 1$$

which you can solve.

In the same way we can find the second column of $[T]^\beta_\alpha$, which gives

$$[T]^\beta_\alpha = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}.$$  

(c) What is the dimension of $\text{Ker}(T)$? Find a basis for $\text{Ker}(T)$.

We put the augmented matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

In reduced row-echelon form, which gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
There are no free variables, and hence \( \ker T = \{ \vec{0} \} \). So \( \ker T \) is zero-dimensional, and has basis \( \{ \vec{0} \} \).

(d) Since the dimension of \( \ker T \) is zero, by the rank-nullity theorem we know that the dimension of the image of \( T \) must equal the dimension of \( \mathbb{R}^2 \), which is two. A basis of the image of the matrix \( [T]_\alpha^\beta \) is

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

To turn this into a basis for the image of \( T \), we need to re-convert these vectors (which are in \( \beta \)-coordinates) into elements of \( P_2 \). Doing so gives the two elements

\[
1 \cdot (x^2 + 2) + 2 \cdot (x^2 + x) + (-1) \cdot 1 = 3x^2 + 2x + 1
\]
\[
-1 \cdot (x^2 + 2) + 0 \cdot (x^2 + x) + (1) \cdot 1 = -x^2 - 1
\]

So \( \{3x^2 + 2x + 1, -x^2 - 1\} \) is a basis for the image of \( T \).