Note: the exam will have 3-4 questions like the ones below, as well as a section of short answer questions similar to the one on the first exam. To review for those, the best thing is to work the true/false review questions listed on the main course webpage.

1. Find a bases for the kernel and image of the following matrix:
   \[ A = \begin{bmatrix}
   1 & 0 & 2 & 0 & 1 \\
   -3 & 0 & 0 & 1 & 0 \\
   -1 & 0 & 4 & 1 & 2 
   \end{bmatrix}. \]

2. Find a basis for the following subspace of \( P_4 \).
   \[ W = \{ p(x) \in P_4 \mid p(1) = p(-1) = 0 \}. \]
   What is the dimension of \( W \)?

3. Determine whether the following mappings are linear transformations. Either prove that a given map is linear or give a counterexample to show it’s not linear.
   (a) \( T : \mathbb{R}^2 \to \mathbb{R}^3 \) defined by \( T((x_1, x_2)) = (2x_1, x_1 + 4, 5x_2) \)
   (b) \( T : P_2 \to P_3 \) defined by \( T(a_2x^2 + a_1x + a_0) = a_0x^3 + (a_1 - a_0)x^2 + 3a_2 - (1/2)a_0 \)

4. Let \( V \) be a subspace of \( \mathbb{R}^n \) with \( \dim(V) = n \). Explain why \( V = \mathbb{R}^n \).

5. (a) Consider the mapping \( T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \) defined by
   \[
   T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & c - b \\ b + 2d - 3c & d + 4a \end{pmatrix}.
   
   Prove that \( T \) is a linear transformation.
   (b) Given the basis \( \alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \) of \( \mathbb{R}^{2\times 2} \),
   give the matrix \( [T]_\alpha \) of \( T \) with respect to the basis \( \alpha \).
   (c) Show that \( T \) is an isomorphism, and use the determinant in your solution.

6. The mapping \( T : \mathbb{R}^2 \to P_2 \) given by \( T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = (a_1 + a_2)x^2 + a_2x + a_1 \) is a linear transformation.
   (a) Prove that \( \alpha = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \) is a basis for \( \mathbb{R}^2 \) and \( \beta = \{x^2 + 2, x^2 + x, 1\} \) is a basis for \( P_2 \).
   (b) Find the matrix \( [T]_\alpha^\beta \)
   (c) What is the dimension of \( \text{Ker}(T) \)? Find a basis for \( \text{Ker}(T) \).
   (d) What is the dimension of \( \text{Im}(T) \)? Find a basis for \( \text{Im}(T) \).