

# Restricted Signed Permutations Counted by the Schröder Numbers\*

Eric S. Egge

Department of Mathematics and Computer Science

Carleton College

Northfield, MN 55057 USA

eggee@member.ams.org

August 24, 2005

## Abstract

Gire, West, and Kremer have found ten classes of restricted permutations counted by the large Schröder numbers, no two of which are trivially Wilf-equivalent. In this paper we enumerate eleven classes of restricted signed permutations counted by the large Schröder numbers, no two of which are trivially Wilf-equivalent. We obtain five of these enumerations by elementary methods, five by displaying isomorphisms with the classical Schröder generating tree, and one by giving an isomorphism with a new Schröder generating tree. When combined with a result of Egge and a routine computer search, this completes the classification of restricted signed permutations counted by the large Schröder numbers in which the set of restrictions consists of two patterns of length 2 and two of length 3.

*Keywords:* Restricted permutation; pattern-avoiding permutation; forbidden subsequence; Schröder number; signed permutation; generating tree

## 1 Introduction and Notation

Let  $B_n$  denote the set of permutations of  $\{1, 2, \dots, n\}$ , written in one-line notation, in which each element may or may not have a bar above it. We refer to the elements of  $B_n$  as *signed permutations*. We write  $S_n$  to denote the set of elements of  $B_n$  with no bars, and we refer to these elements as *classical permutations*. For any signed permutation  $\pi$  we write  $|\pi|$  to denote the length of  $\pi$  and we write  $||\pi||$  to denote the classical permutation obtained by removing the bars from  $\pi$ . For a given signed permutation  $\pi$  we write  $\pi(i)$  to denote the  $i$ th entry of  $\pi$ .

Suppose  $\pi$  and  $\sigma$  are signed permutations. We say a subsequence of  $\pi$  has *type*  $\sigma$  whenever it has all of the same pairwise comparisons as  $\sigma$  and an entry in the subsequence of  $\pi$  is barred

---

\*2000 Mathematics Subject Classification: Primary 05A05, 05A15

if and only if the corresponding entry in  $\sigma$  is barred. For example, the subsequence  $3\bar{4}71$  of the signed permutation  $93\bar{4}72\bar{8}\bar{5}16$  has type  $2\bar{3}41$ . We say  $\pi$  *avoids*  $\sigma$  whenever  $\pi$  has no subsequence of type  $\sigma$ . For example, the signed permutation  $93\bar{4}72\bar{8}\bar{5}16$  avoids  $\bar{3}2\bar{1}$  and  $1432$  but it has  $92\bar{5}$  as a subsequence so it does not avoid  $31\bar{2}$ . In this setting  $\sigma$  is sometimes called a *pattern* or a *forbidden subsequence* and  $\pi$  is sometimes called a *restricted permutation* or a *pattern-avoiding permutation*. In this paper we will be interested in signed permutations which avoid several patterns, so for any set  $R$  of signed permutations we write  $B_n(R)$  to denote the set of signed permutations of length  $n$  which avoid every pattern in  $R$  and we write  $B(R)$  to denote the set of all signed permutations which avoid every pattern in  $R$ . When  $R = \{\pi_1, \dots, \pi_r\}$  we often write  $B_n(R) = B_n(\pi_1, \dots, \pi_r)$  and  $B(R) = B(\pi_1, \dots, \pi_r)$ . When we wish to discuss classical permutations, we replace  $B$  with  $S$  in the above notation.

Suppose  $R_1$  and  $R_2$  are sets of signed permutations. We say  $R_1$  and  $R_2$  are *Wilf-equivalent* whenever  $|B_n(R_1)| = |B_n(R_2)|$  for all  $n \geq 0$ . There are four natural operations which preserve Wilf-equivalence classes:

- the bar operator, which replaces each barred entry in a signed permutation with its unbarred counterpart and vice versa;
- the reverse operator, which writes the entries of a signed permutation in reverse order;
- the complement operator, which replaces each entry  $\pi(i)$  of a signed permutation  $\pi$  with  $|\pi| + 1 - \pi(i)$ ;
- the inverse operator, which takes replaces each signed permutation with its group-theoretic inverse.

For example, if  $\pi = 2\bar{4}13$  then  $bar(\pi) = \bar{2}4\bar{1}\bar{3}$ ,  $reverse(\pi) = 31\bar{4}2$ ,  $complement(\pi) = 3\bar{1}42$ , and  $inverse(\pi) = 314\bar{2}$ . If we write each signed permutation as a square permutation matrix in which barred entries are represented by  $-1$ s, then the bar operator is multiplication by  $-1$ , the reverse operator is the reflection over the vertical axis, the complement operator is the reflection over the horizontal axis, and the inverse operator is both the reflection over the main diagonal and the usual matrix inverse. From this one can show that the group  $G$  generated by these operations is isomorphic to  $D_8 \oplus Z_2$ , where  $D_8$  is the dihedral group of order 8. We say  $R_1$  is *trivially Wilf-equivalent* to  $R_2$  whenever  $R_2$  is the image of  $R_1$  under some element of  $G$ .

The focus of this paper is on restricted signed permutations, but it includes a major role for the large Schröder numbers. (There are also small Schröder numbers, which are, up to a shift in index, half of the large Schröder numbers.) The *large Schröder numbers* (hereafter just the Schröder numbers) may be recursively defined by  $r_0 = 1$  and  $r_n = r_{n-1} + \sum_{k=1}^n r_{k-1}r_{n-k}$  for  $n \geq 1$ . From this definition one can show that the generating function for the Schröder numbers is given by

$$\sum_{n=0}^{\infty} r_n x^n = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}. \quad (1)$$

The Schröder numbers are closely related to the ubiquitous Catalan numbers, and in particular one can also show that

$$r_n = \sum_{d=0}^n \binom{2n-d}{d} C_{n-d} \quad (n \geq 0). \quad (2)$$

Here  $C_n$  is the  $n$ th Catalan number, which may be defined by  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . For a list of some of the known combinatorial interpretations of the Schröder numbers, see [10, pp. 239–240].

Gire [5], West [11], and Kremer [6] have found ten classes of classical pattern-avoiding permutations which are enumerated by the Schröder numbers. (See [2] for other work along these lines.) Each of the corresponding sets of forbidden patterns consists of two classical permutations, each of length 4, and no two of these sets are trivially Wilf-equivalent. A computer search reveals that every pair of classical permutations of length 4 whose pattern-avoiding permutations are enumerated by Schröder numbers is trivially Wilf-equivalent to one of the ten sets found by Gire, West, and Kremer, but to the best of our knowledge it is not known whether there are other classes of classical pattern-avoiding permutations which are enumerated by the Schröder numbers.

The enumeration of restricted signed permutations was introduced by Simion [9] and studied further by Mansour and West [7]. In this paper we continue this study, considering in particular the following eleven sets of forbidden signed permutations.

- |  |   |   |
|--|---|---|
| I. $\overline{21}, \overline{21}, \overline{123}, 123$   | V. $\overline{21}, \overline{21}, \overline{123}, 321$    | IX. $21, 2\overline{1}, \overline{231}, \overline{312}$ |
| II. $\overline{21}, \overline{21}, \overline{123}, 132$  | VI. $21, 2\overline{1}, \overline{312}, \overline{312}$   | X. $21, 2\overline{1}, \overline{132}, \overline{132}$  |
| III. $\overline{21}, \overline{21}, \overline{123}, 231$ | VII. $21, 2\overline{1}, \overline{321}, \overline{321}$  | XI. $21, \overline{21}, \overline{321}, \overline{312}$ |
| IV. $\overline{21}, \overline{21}, \overline{123}, 312$  | VIII. $21, 2\overline{1}, \overline{321}, \overline{312}$ |   |

We begin by using elementary methods to show that if  $T$  is any set of classical permutations then  $|B_n(\overline{21}, \overline{21}, \overline{123}, T)|$  is a convolution of  $|S_n(T)|$  with a certain sequence of binomial coefficients. Combining this with (2), we conclude that for any set  $R$  among I–V we have  $|B_n(R)| = r_n$  for all  $n \geq 0$ . Next we recall some basic facts concerning generating trees, including the classical Schröder tree, and we describe how each set of restricted signed permutations can be organized into a generating tree in a natural way. For each set  $R$  among VI–X we give an isomorphism between the generating tree for  $B(R)$  and the classical Schröder tree. It follows that for each set  $R$  among VI–X we have  $|B_n(R)| = r_n$  for all  $n \geq 0$ . We then introduce a new generating tree, which we call the tilted Schröder tree. We use the kernel method to prove that for all  $n \geq 0$  this tree has exactly  $r_n$  nodes on level  $n$ , and we give an isomorphism between the generating tree for  $B(21, \overline{21}, \overline{321}, \overline{312})$  and the tilted Schröder tree. It follows that  $|B_n(21, \overline{21}, \overline{321}, \overline{312})| = r_n$  for all  $n \geq 0$ .

EGge has previously shown [4] that  $|B_n(2\overline{1}, \overline{21}, \overline{312}, 312)| = r_n$  for all  $n \geq 0$ . It is routine to check that no two of the sets of forbidden signed permutations in I–XI are trivially Wilf-equivalent, and that none is trivially Wilf-equivalent to  $2\overline{1}, \overline{21}, \overline{312}, 312$ . A computer search reveals that if  $R$  is any set of signed permutations consisting of two signed permutations of length 2 and two of length 3, and  $|B_n(R)| = r_n$  for  $0 \leq n \leq 5$ , then  $R$  is trivially

Wilf-equivalent to  $\overline{21}, \overline{312}, 312$  or to one of the sets in I–XI. In short, up to trivial Wilf-equivalence, this paper completes the classification of restricted signed permutations counted by the Schröder numbers whose forbidden patterns consist of two patterns of length 2 and two of length 3. To the best of our knowledge, it is not known whether there are other classes of restricted signed permutations counted by the Schröder numbers.

## 2 A Convolution with Certain Binomial Coefficients

Some enumerations of pattern-avoiding signed permutations can be obtained from corresponding enumerations of classical pattern-avoiding permutations. For instance, the fact that  $|B_n| = 2^n |S_n|$  for  $n \geq 0$  can be generalized as follows.

**Definition 2.1** *For any set  $T$  of classical permutations, we write  $\hat{T}$  to denote the set of signed permutations obtained by putting bars over the entries in the elements of  $T$  in all possible ways.*

To illustrate Definition 2.1, we observe that if  $T = \{12\}$  then  $\hat{T} = \{12, \overline{12}, \overline{1\overline{2}}, \overline{\overline{12}}\}$ .

**Proposition 2.2** *For any set  $T$  of classical permutations, we have*

$$B_n(\hat{T}) = \widehat{S_n(T)} \quad (n \geq 0). \quad (3)$$

*In particular,*

$$|B_n(\hat{T})| = 2^n |S_n(T)| \quad (n \geq 0). \quad (4)$$

*Proof.* To prove (3), note that  $\pi \in B_n(\hat{T})$  if and only if  $|\pi| \in S_n(T)$ .

Line (4) is immediate from (3).  $\square$

Proposition 2.2 allows us to recover a result of Mansour and West.

**Corollary 2.3** *([7, Eq. (4.4)]) For all  $n \geq 0$  we have  $|B_n(12, \overline{12}, \overline{1\overline{2}}, \overline{\overline{12}})| = 2^n$ .*

*Proof.* Set  $T = \{12\}$  in (4) and use the fact that  $|S_n(12)| = 1$  for all  $n \geq 0$ .  $\square$

The main result of this section relates enumerations of classical pattern-avoiding permutations with pattern-avoiding signed permutations by a convolution with certain binomial coefficients. To prove the result, we'll use the following technical lemma.

**Lemma 2.4** *A signed permutation  $\pi$  avoids  $\overline{21}, \overline{2\overline{1}}$ , and  $1\overline{2}3$  if and only if both of the following hold.*

- (i) *The barred entries of  $\pi$  are in increasing order.*
- (ii) *If a barred entry of  $\pi$  has an unbarred entry to its right, then the barred entry is less than all unbarred entries of  $\pi$ .*

*Proof.* ( $\implies$ ) Suppose  $\pi$  avoids  $\overline{21}, \overline{2\overline{1}}$ , and  $1\overline{2}3$ . First observe that (i) follows from the fact that  $\pi$  avoids  $\overline{21}$ . To prove (ii), suppose  $\pi(i)$  is barred,  $\pi(j)$  is unbarred, and  $i < j$ . If there is an unbarred entry  $a$  of  $\pi$  to the right of  $\pi(i)$  such that  $\pi(i) > a$  then  $\pi(i)a$  has type  $\overline{21}$ . If there is an unbarred entry  $a$  of  $\pi$  to the left of  $\pi(i)$  such that  $\pi(i) > a$  then  $a\pi(i)\pi(j)$  has type  $1\overline{2}3$ . Now (ii) follows.

( $\Leftarrow$ ) Suppose (i) and (ii) hold for a signed permutation  $\pi$ . By (i),  $\pi$  avoids  $\overline{21}$ . If  $\pi$  contains a pattern of type  $\overline{21}$  or  $1\overline{23}$  then the entry playing the role of the  $\overline{2}$  violates (ii), so  $\pi$  avoids  $\overline{21}$  and  $1\overline{23}$ .  $\square$

In our next result, we use Lemma 2.4 to count signed permutations which have a given number of barred entries and which avoid  $\overline{21}$ ,  $\overline{21}$ ,  $1\overline{23}$ , and any set  $T$  of classical patterns.

**Proposition 2.5** *Let  $T$  denote a set of classical permutations. Then for all  $n \geq 0$  and all  $d$  such that  $0 \leq d \leq n$  there are  $\binom{2n-d}{d} |S_{n-d}(T)|$  signed permutations in  $B_n$  which avoid  $\overline{21}$ ,  $\overline{21}$ ,  $1\overline{23}$ , and  $T$  and which have exactly  $d$  barred entries.*

*Proof.* Fix  $n \geq 0$  and  $d$  such that  $0 \leq d \leq n$ . We first describe an algorithm for constructing a signed permutation  $\pi$  which avoids  $\overline{21}$ ,  $\overline{21}$ ,  $1\overline{23}$ , and  $T$  and which has exactly  $d$  barred entries.

1. Choose a classical permutation  $\sigma \in S_{n-d}(T)$ .
2. Construct the graph of  $\sigma$  by placing dots at the points  $(i, \sigma(i))$  for  $1 \leq i \leq n-d$ .
3. View the  $n-d$  vertical lines  $x = \frac{1}{2}, x = \frac{3}{2}, \dots, x = n-d - \frac{1}{2}$  and the  $n-d+1$  horizontal lines  $y = \frac{1}{2}, y = \frac{3}{2}, \dots, y = n-d + \frac{1}{2}$  as baskets, and place a total of  $d$  indistinguishable balls in the  $2n-2d+1$  baskets.
4. Produce the graph of a signed permutation as follows.
  - (a) Space the points of the graph of  $\sigma$  and the inserted balls in order horizontally, starting with the balls on  $x = \frac{1}{2}$ , followed by the dot at  $(1, \sigma(1))$ , followed by the balls on  $x = \frac{3}{2}$ , etc., followed by the dot at  $(n-d, \sigma(n-d))$ , followed by the balls on  $y = \frac{1}{2}$ , followed by the balls on  $y = \frac{3}{2}$ , etc.
  - (b) Space the points of the graph of  $\sigma$  and the inserted balls in order vertically, starting with the balls on  $x = \frac{1}{2}$ , followed by the balls on  $x = \frac{3}{2}$ , etc., followed by the balls on  $x = n-d - \frac{1}{2}$ , followed by the balls on  $y = \frac{1}{2}$ , followed by the dot at  $(\sigma^{-1}(1), 1)$ , followed by the balls on  $y = \frac{3}{2}$ , followed by the dot at  $(\sigma^{-1}(2), 2)$ , etc.

Observe that step 4 guarantees that this algorithm always produces (the graph of) a signed permutation of length  $n$  with exactly  $d$  barred entries. In view of Lemma 2.4, the algorithm produces only signed permutations which avoid  $\overline{21}$ ,  $\overline{21}$ ,  $1\overline{23}$ , and  $T$ , and each such signed permutation is produced in exactly one way. Steps 2 and 4 may each be performed in just one way, and step 1 may be performed in  $|S_{n-d}(T)|$  ways. In step 3 we are inserting  $d$  indistinguishable balls in  $2n-2d+1$  distinguishable baskets; there are  $\binom{2n-d}{d}$  ways to do this. The result follows.  $\square$

Proposition 2.5 gives us two sets of signed permutations counted by the Fibonacci numbers, one of which was previously found by Mansour and West [7, Eq. (3.5)] and later by Egge [4].

**Proposition 2.6** *For any permutation  $\sigma \in S_2$  we have*

$$|B_n(\overline{21}, \overline{21}, 1\overline{23}, \sigma)| = F_{2n+1} \quad (n \geq 0), \quad (5)$$

where  $F_n$  is the  $n$ th Fibonacci number, defined by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ .

*Proof.* Set  $T = \{\sigma\}$  in Proposition 2.5 and use the fact that  $|S_n(T)| = 1$  for all  $n \geq 0$  to obtain

$$|B_n(\overline{21}, \overline{21}, 1\overline{23}, \sigma)| = \sum_{d=0}^n \binom{2n-d}{d} \quad (n \geq 0).$$

This last expression is well-known to be equal to  $F_{2n+1}$ . (See [3, Thm. 7.1.2], for instance.)  $\square$

Proposition 2.5 gives us five new sets of signed permutations counted by the Schröder numbers.

**Theorem 2.7** *For any permutation  $\sigma \in S_3$  we have*

$$|B_n(\overline{21}, \overline{21}, 1\overline{23}, \sigma)| = r_n \quad (n \geq 0). \quad (6)$$

We observe that the sets corresponding to  $\sigma = 132$  and  $\sigma = 213$  are trivially Wilf-equivalent.

*Proof.* Set  $T = \{\sigma\}$  in Proposition 2.5, use the fact that  $|S_n(\sigma)| = C_n$  for  $n \geq 0$ , and use (2) to simplify the result.  $\square$

### 3 An Interlude on Generating Trees

For our purposes, a *generating tree* is a rooted, labeled tree in which the label of each node determines that node's number of children and their labels. We generally specify a particular generating tree by giving two pieces of data:

- the label of the root node;
- a list of succession rules, which state for each label  $k$  the number of children a node with label  $k$  has and what their labels are.

Many generating trees of combinatorial significance are known; [1, 12] contain several examples of interest. We content ourselves here by recalling one which is particularly relevant.

**Example 3.1** [12, Ex. 5] *The classical Schröder generating tree is given by*

- *Root:* (2);
- *Rule:*  $(k) \rightarrow (3)(4) \cdots (k-1)(k)(k+1)(k+1)$  for  $k \geq 2$ .

Given a generating tree, we are often interested in how many nodes it has on level  $n$ , where the root is on level 0. The classical Schröder tree gets its name from the well-known fact that it has  $r_n$  nodes on level  $n$  for all  $n \geq 0$ .

Signed pattern-avoiding permutations (as well as classical pattern-avoiding permutations) have a natural generating tree structure. To describe this structure, suppose  $T$  is a set of forbidden signed permutations. The nodes on level  $n$  of the associated generating tree are the elements of  $B_n(T)$ , and in the absence of a simpler labeling scheme, we regard each

node as being labeled with its associated signed permutation. Each signed permutation  $\pi \in B_{n-1}(T)$  has  $n$  spaces in which an  $n$  or an  $\bar{n}$  may be inserted to produce a signed permutation in  $B_n$ , but in general only some of the resulting signed permutations will avoid  $T$ . For us the children of  $\pi \in B_{n-1}(T)$  are those signed permutations which are obtained from  $\pi$  by inserting  $n$  or  $\bar{n}$  and which avoid  $T$ . Given  $\pi \in B_{n-1}(T)$ , we call an insertion space *unbar-active* (resp. *bar-active*) whenever insertion of  $n$  (resp.  $\bar{n}$ ) in that space produces an element of  $B_n(T)$  and we call the space *unbar-inactive* (resp. *bar-inactive*) otherwise.

**Example 3.2** *Suppose  $T = \{\bar{2}1, 123\}$  and  $\pi = 4\bar{1}52\bar{3}$ . Then the left-most three spaces of  $\pi$  are unbar-active, the remaining spaces are unbar-inactive, the right-most two spaces of  $\pi$  are bar-active, and the remaining spaces are bar-inactive.*

Suppose  $T$  is a set of forbidden patterns and  $\pi \in B_{n-1}(T)$  for some  $n \geq 1$ . We note that if a space is unbar-inactive in  $\pi$  then it retains that status in all of the children of  $\pi$ . Inserting  $n$  into an unbar-inactive space of  $\pi$  produces a forbidden pattern, and since  $\pi$  avoids  $T$ , the newly inserted entry  $n$  must participate in that forbidden pattern. Along the same lines, suppose spaces  $s_1$  and  $s_2$  in  $\pi$  are unbar-active, but insertion of  $n$  in  $s_1$  causes  $s_2$  to become unbar-inactive. Then the signed permutation obtained from  $\pi$  by inserting  $n$  in  $s_1$  and  $n+1$  in  $s_2$  contains a forbidden pattern, and  $n$  and  $n+1$  both participate in that pattern. Similar comments hold for bar-active and bar-inactive spaces.

## 4 Isomorphisms to the Classical Schröder Tree

In this section we consider the following five sets of forbidden signed permutations.

- $T_1 = \{21, 2\bar{1}, \bar{3}1\bar{2}, \bar{3}1\bar{2}\}$
- $T_2 = \{21, 2\bar{1}, \bar{3}2\bar{1}, \bar{3}2\bar{1}\}$
- $T_3 = \{21, 2\bar{1}, \bar{3}2\bar{1}, \bar{3}1\bar{2}\}$
- $T_4 = \{21, 2\bar{1}, \bar{2}3\bar{1}, \bar{3}1\bar{2}\}$
- $T_5 = \{21, 2\bar{1}, \bar{1}3\bar{2}, \bar{1}3\bar{2}\}$

Following [6, 11], for each of  $T_1 - T_5$  we give an isomorphism between the generating tree for the associated pattern-avoiding signed permutations and the classical Schröder generating tree. To obtain our isomorphisms, we analyze the behavior of the unbar-active and bar-active spaces of a signed permutation when  $n$  or  $\bar{n}$  is inserted. We begin by observing that if  $\pi$  avoids any of  $T_1 - T_5$  then just one space in  $\pi$  can be unbar-active.

**Lemma 4.1** *Suppose  $n \geq 2$  and  $T$  is one of  $T_1 - T_5$  above. Fix  $\pi \in B_{n-1}(T)$ . Then the right-most space of  $\pi$  is unbar-active and all other spaces of  $\pi$  are unbar-inactive.*

*Proof.* Observe that no forbidden pattern ends with its largest entry, so the right-most space of  $\pi$  is unbar-active. However, inserting  $n$  into any other space will produce a subsequence of type 21 or a subsequence of type  $2\bar{1}$ , depending on whether the right-most entry of  $\pi$  is barred. Both patterns are forbidden, so no other space in  $\pi$  is unbar-active.  $\square$

Next we analyze the effect inserting  $n$  has on the bar-active spaces.

**Lemma 4.2** *Suppose  $n \geq 2$  and  $T$  is one of  $T_1 - T_5$  above. Fix  $\pi \in B_{n-1}(T)$  and let  $\pi^+$  denote the signed permutation obtained by appending  $n$  to the right end of  $\pi$ . Then the following hold.*

- (i) *The right-most two spaces of  $\pi^+$  are bar-active.*
- (ii) *Suppose  $s$  is a space in  $\pi^+$  which is not one of the right-most two spaces. Abusing notation, we identify  $s$  with the corresponding space in  $\pi$ . Then  $s$  is bar-active in  $\pi^+$  if and only if it is bar-active in  $\pi$ .*

*Proof.* (i) None of the forbidden patterns of length 2 contains  $\bar{2}$  and none of the forbidden patterns of length 3 ends with  $\bar{3}$ , so the right-most space of  $\pi^+$  is bar-active. Similarly, none of the forbidden patterns of length 3 contains  $\bar{3}2$ , so the second space from the right end of  $\pi^+$  is bar-active.

(ii) If  $s$  is bar-inactive in  $\pi$  then it remains so in  $\pi^+$ , so suppose by way of contradiction that  $s$  is bar-active in  $\pi$  and bar-inactive in  $\pi^+$ . Then inserting  $\overline{n+1}$  in  $\pi^+$  at  $s$  produces a forbidden pattern in which both  $n$  and  $\overline{n+1}$  participate. But none of the forbidden patterns of length 2 contains  $\bar{2}$  and none of the forbidden patterns of length 3 contains 2. This is a contradiction, so  $s$  must be bar-active in  $\pi^+$ .  $\square$

We now give an isomorphism between the generating tree for  $B(T_1)$  and the classical Schröder generating tree. This example illustrates one of the simpler ways the classical Schröder tree can appear as a tree of pattern-avoiding permutations.

**Theorem 4.3** *For each signed permutation  $\pi \in B(T_1)$ , let  $f_1(\pi)$  denote one plus the number of spaces in  $\pi$  with no subsequence of type  $\bar{1}2$  or  $1\bar{2}$  to their right. Then the map  $\pi \mapsto f_1(\pi)$  is an isomorphism of generating trees between the generating tree for  $B(T_1)$  and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\bar{1}, \bar{3}1\bar{2}, \bar{3}1\bar{2})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $f_1(\emptyset) = 2$  and  $f_1(1) = f_1(\bar{1}) = 3$ , so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_1)$ . Observe that since  $\bar{2}$  does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are  $\bar{3}1\bar{2}$  and  $\bar{3}1\bar{2}$ , a space in  $\pi$  is bar-active if and only if it has no subsequence of type  $\bar{1}2$  or  $1\bar{2}$  to its right. In view of Lemma 4.1, the signed permutation  $\pi$  has  $f_1(\pi) - 1$  spaces which are bar-active and 1 space which is unbar-active, so  $\pi$  has  $f_1(\pi)$  children. In view of Lemma 4.2, the child of  $\pi$  obtained by inserting  $n$  has  $f_1(\pi) + 1$  children, so its image under  $f_1$  is  $f_1(\pi) + 1$ . To count the children of the signed permutations obtained by inserting  $\bar{n}$  into  $\pi$ , observe that the inserted  $\bar{n}$  and the entry immediately to its left form a subsequence of type  $\bar{1}2$  or  $1\bar{2}$ . Therefore the bar-active spaces in the new signed permutation are the spaces to the right of  $\bar{n}$  and the space immediately to the left of  $\bar{n}$ . Hence the children of  $\pi$  have  $3, 4, \dots, f_1(\pi), f_1(\pi) + 1, f_1(\pi) + 1$  children, and the result follows.  $\square$

Next we describe an isomorphism between the generating tree for  $B(T_2)$  and the classical Schröder generating tree. Here the classical Schröder tree appears in a slightly more complicated way.

**Theorem 4.4** *For each signed permutation  $\pi \in B(T_2)$ , let  $f_2(\pi)$  denote one plus the number of spaces in  $\pi$  with no subsequence of type  $\bar{2}1$  or  $\bar{2}1$  to their right. Then the map  $\pi \mapsto f_2(\pi)$*

is an isomorphism of generating trees between the generating tree for  $B(T_2)$  and the classical Schröder generating tree. In particular,

$$|B_n(21, 2\bar{1}, \overline{321}, \overline{321})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $f_2(\emptyset) = 2$  and  $f_2(1) = f_2(\bar{1}) = 3$ , so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_2)$ . Observe that since  $\bar{2}$  does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are  $\overline{321}$  and  $\overline{321}$ , a space in  $\pi$  is bar-active if and only if it has no subsequence of type  $\overline{21}$  or  $\bar{2}1$  to its right. In view of Lemma 4.1, the signed permutation  $\pi$  has  $f_2(\pi) - 1$  spaces which are bar-active and 1 space which is unbar-active, so  $\pi$  has  $f_2(\pi)$  children. In view of Lemma 4.2, the child of  $\pi$  obtained by inserting  $n$  has  $f_2(\pi) + 1$  children, so its image under  $f_2$  is  $f_2(\pi) + 1$ . To count the children of the signed permutations obtained by inserting  $\bar{n}$  into  $\pi$ , first observe that if  $\bar{n}$  is inserted in the right-most space, all bar-active spaces remain bar-active. If  $\bar{n}$  is inserted in any other space, it forms a subsequence of type  $\overline{21}$  or  $\bar{2}1$  with the entry immediately to its right. In this case all spaces to the left of  $\bar{n}$  become bar-inactive, while those to the right of  $\bar{n}$  remain bar-active. Therefore the children of  $\pi$  have  $f_2(\pi) + 1, 3, 4, \dots, f_2(\pi), f_2(\pi) + 1$  children, and the result follows.  $\square$

The generating tree for  $B(T_3)$  is the classical Schröder tree in an even more complicated way.

**Theorem 4.5** *For each signed permutation  $\pi \in B(T_3)$ , let  $f_3(\pi)$  denote one plus the number of spaces in  $\pi$  with no subsequence of type  $\overline{21}$  or  $1\bar{2}$  to their right. Then the map  $\pi \mapsto f_3(\pi)$  is an isomorphism of generating trees between the generating tree for  $B(T_3)$  and the classical Schröder generating tree. In particular,*

$$|B_n(21, 2\bar{1}, \overline{321}, \overline{312})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $f_3(\emptyset) = 2$ , and  $f_3(1) = f_3(\bar{1}) = 3$ , so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_3)$ . Observe that since  $\bar{2}$  does not appear in the forbidden patterns of length 2, and since the other two forbidden patterns are  $\overline{321}$  and  $\overline{312}$ , a space in  $\pi$  is bar-active if and only if it has no subsequence of type  $\overline{21}$  or  $1\bar{2}$  to its right. In view of Lemma 4.1, the signed permutation  $\pi$  has  $f_3(\pi) - 1$  spaces which are bar-active and 1 space which is unbar-active, so  $\pi$  has  $f_3(\pi)$  children. In view of Lemma 4.2, the child of  $\pi$  obtained by inserting  $n$  has  $f_3(\pi) + 1$  children, so its image under  $f_3$  is  $f_3(\pi) + 1$ . To count the children of the signed permutations obtained by inserting  $\bar{n}$  into  $\pi$ , first observe that all of the spaces to the right of the left-most bar-active space are also bar-active, and that the subsequence  $\sigma$  enclosed by the bar-active spaces avoids  $21$ ,  $2\bar{1}$ ,  $\bar{2}1$ , and  $1\bar{2}$ . Hence  $\sigma$  consists of a sequence of barred entries, in increasing order, followed by a sequence of unbarred entries, in increasing order. We consider three cases.

**Case One:**  $\sigma$  has no barred entries.

Observe that when we insert  $\bar{n}$  into  $\sigma$ , we create a pattern of type  $1\bar{2}$  consisting of  $\bar{n}$  and the necessarily unbarred entry immediately to its left. It follows that the bar-active spaces in

the new signed permutation are those to the right of  $\bar{n}$  and the space immediately to the left. Therefore the children of  $\pi$  obtained by inserting  $\bar{n}$  have  $3, 4, \dots, f_3(\pi), f_3(\pi) + 1, f_3(\pi) + 1$  children.

**Case Two:**  $\sigma$  has no unbarred entries.

First observe that when we insert  $\bar{n}$  into the right-most space in  $\sigma$ , the spaces immediately left and right of  $\bar{n}$  are bar-active. Moreover, since no bar-active space in  $\pi$  has an unbarred entry to its right and the only forbidden pattern which ends with its second largest entry barred is  $\bar{3}1\bar{2}$ , all spaces which are bar-active in  $\pi$  are bar-active in the new signed permutation.

Now suppose we insert  $\bar{n}$  into a bar-active space other than the right-most space in  $\sigma$ . In this case we create a pattern of type  $\bar{2}\bar{1}$  consisting of  $\bar{n}$  and the entry immediately to its right. Therefore the bar-active spaces in the new signed permutation are those to the right of  $\bar{n}$ . It follows that the children of  $\pi$  obtained by inserting  $\bar{n}$  have  $f_3(\pi) + 1, 3, 4, \dots, f_3(\pi), f_3(\pi) + 1$  children.

**Case Three:**  $\sigma$  has both barred and unbarred entries.

Observe that when we insert  $\bar{n}$  into the bar-active space between the last barred entry and the first unbarred entry in  $\sigma$  we create no subsequence of type  $\bar{2}\bar{1}$  or  $1\bar{2}$  in  $\sigma$ , so the resulting signed permutation has  $f_3(\pi) + 1$  children. Now suppose we insert  $\bar{n}$  among the unbarred entries of  $\sigma$ . Then  $\bar{n}$ , together with the unbarred entry immediately to its left, forms a  $1\bar{2}$  pattern. It follows that the bar-active spaces in the new signed permutation are those to the right of  $\bar{n}$  and the space immediately to the left of  $\bar{n}$ . Finally, suppose we insert  $\bar{n}$  among the barred entries of  $\sigma$ . Then  $\bar{n}$ , together with the barred entry immediately to its right, forms a  $\bar{2}\bar{1}$  pattern. It follows that the bar-active spaces in the new signed permutation are those to the right of  $\bar{n}$ . Combining these observations, we find that the children of  $\pi$  obtained by inserting  $\bar{n}$  have  $3, 4, \dots, k + 2, f_3(\pi) + 1, k + 3, \dots, f_3(\pi), f_3(\pi) + 1$  children, where  $k$  is the number of unbarred entries in  $\sigma$ .

In all cases the children of  $\pi$  have  $3, 4, \dots, f_3(\pi), f_3(\pi) + 1, f_3(\pi) + 1$  children, and the result follows.  $\square$

Next we describe an isomorphism between the generating tree for  $B(T_4)$  and the classical Schröder generating tree. In contrast to our previous three isomorphisms, this map does not include a detailed description of the locations of the bar-active spaces in a given permutation. Nevertheless, the proof of the theorem restricts these locations somewhat.

**Theorem 4.6** *For each signed permutation  $\pi \in B(T_4)$ , let  $f_4(\pi)$  denote one plus the number of bar-active spaces in  $\pi$ . Then the map  $\pi \mapsto f_4(\pi)$  is an isomorphism of generating trees between the generating tree for  $B(T_4)$  and the classical Schröder generating tree. In particular,*

$$|B_n(21, \bar{2}\bar{1}, \bar{2}\bar{3}\bar{1}, \bar{3}\bar{1}\bar{2})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $f_4(\emptyset) = 2$  and  $f_4(1) = f_4(\bar{1}) = 3$ , so both trees have the same first two levels. In view of Lemma 4.1, the quantity  $f_4(\pi)$  is the number of children of  $\pi$ , so it is sufficient to verify these children have the correct labels.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_4)$ . We consider two cases.

**Case One:**  $\pi$  ends with a (possibly empty) sequence  $\sigma_u$  of unbarred entries, which is immediately preceded by a nonempty sequence  $\sigma_b$  of barred entries.

We first claim that no space to the left of the right-most entry left of  $\sigma_b$ , which we denote by  $a$ , is bar-active. To see this, first observe that  $a$  and the barred entry to its right form a sequence of type  $2\bar{1}$  or  $\bar{12}$ . The first case is forbidden. If we insert  $\bar{n}$  to the left of the right-most unbarred entry in the second case we produce a sequence of type  $\bar{3}1\bar{2}$ , which is forbidden, and the claim follows.

Suppose we insert  $\bar{n}$  into a bar-active space in  $\sigma_u$ , so that  $\bar{n}$  has an unbarred entry to its left. The only forbidden pattern of length 3 which has  $\bar{2}$  and  $\bar{3}$  adjacent is  $\bar{2}\bar{3}\bar{1}$  and no entry to the right of  $\bar{n}$  in the new signed permutation is barred, so the spaces immediately left and right of  $\bar{n}$  are bar-active in the new signed permutation. Since  $\bar{3}1\bar{2}$  is forbidden, all other spaces to the left of  $\bar{n}$  are bar-inactive in the new signed permutation. Since no forbidden pattern of length 3 has 1,  $\bar{2}$ , and  $\bar{3}$  with  $\bar{2}$  and  $\bar{3}$  in increasing order, all spaces to the right of  $\bar{n}$  which were bar-active in  $\pi$  are bar-active in the new signed permutation.

Now suppose we insert  $\bar{n}$  in the space between  $\sigma_b$  and  $\sigma_u$ . As above, the spaces immediately left and right of  $\bar{n}$  are bar-active in the new signed permutation. Moreover, if a space was bar-active in  $\pi$  then it is also bar-active in the new signed permutation, since inserting  $\bar{n} + \bar{1}$  in  $\sigma_b$  cannot create a subsequence of type  $\bar{3}1\bar{2}$  and inserting  $\bar{n} + \bar{1}$  in  $\sigma_u$  cannot create a subsequence of type  $\bar{2}\bar{3}\bar{1}$ .

Finally, suppose we insert  $\bar{n}$  in a bar-active space in  $\sigma_b$ , so that  $\bar{n}$  has a barred entry to its right. As above, the space immediately left of  $\bar{n}$  is bar-active, but the space immediately right of  $\bar{n}$  is not, since inserting  $\bar{n} + \bar{1}$  in that space produces a  $\bar{2}\bar{3}\bar{1}$  pattern, which is forbidden. Similarly, every space to the right of  $\bar{n}$  in  $\sigma_b$  is bar-inactive in the new signed permutation. All other spaces which were bar-active in  $\pi$  remain bar-active in the new signed permutation, since inserting  $\bar{n} + \bar{1}$  to the left of  $\bar{n}$  cannot create a subsequence of type  $\bar{3}1\bar{2}$  and inserting  $\bar{n} + \bar{1}$  in  $\sigma_u$  cannot create a subsequence of type  $\bar{2}\bar{3}\bar{1}$ .

Combining the observations above, and using the fact that the space between  $\sigma_b$  and  $\sigma_u$  is always bar-active, we find the children of  $\pi$  have  $3, 4, \dots, k + 2, f_4(\pi) + 1, f_4(\pi), \dots, k + 3, f_4(\pi) + 1$  children, where  $k$  is the number of bar-active spaces in  $\sigma_u$ .

**Case Two:** All entries in  $\pi$  are unbarred.

First observe that since  $\pi$  avoids 21, we have  $\pi = 12 \cdots n - 1$ , and every space is bar-active. Now suppose we insert  $\bar{n}$ . As in Case One, the spaces immediately left and right of  $\bar{n}$  are bar-active in the new signed permutation, the rest of the spaces to the left of  $\bar{n}$  are bar-inactive, and all spaces to the right of  $\bar{n}$  are bar-active. Therefore the children of  $\pi$  have  $3, 4, \dots, f_4(\pi), f_4(\pi) + 1, f_4(\pi) + 1$  children.

Observe that in both cases the children of  $\pi$  have  $3, 4, \dots, f_4(\pi), f_4(\pi) + 1, f_4(\pi) + 1$  children, and the result follows.  $\square$

We conclude this section by describing an isomorphism between the generating tree for  $B(T_5)$  and the classical Schröder generating tree.

**Theorem 4.7** *For each signed permutation  $\pi \in B(T_5)$ , let  $f_5(\pi)$  denote one plus the number of bar-active spaces in  $\pi$ . Then the map  $\pi \mapsto f_5(\pi)$  is an isomorphism of generating trees between the generating tree for  $B(T_5)$  and the classical Schröder generating tree. In*

particular,

$$|B_n(21, 2\bar{1}, \overline{132}, \overline{13\bar{2}})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $f_5(\emptyset) = 2$  and  $f_5(1) = f_5(\bar{1}) = 3$ , so both trees have the same first two levels. In view of Lemma 4.1, the quantity  $f_5(\pi)$  is the number of children of  $\pi$ , so it is sufficient to verify these children have the correct labels.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_5)$ . First observe that since no forbidden pattern begins with  $\bar{2}$  or  $\bar{3}$ , the left-most space in  $\pi$  is bar-active. Suppose we insert  $\bar{n}$  into the left-most space of  $\pi$ . Since  $\bar{2}$  does not appear before  $\bar{3}$  in any forbidden pattern, the spaces immediately left and right of  $\bar{n}$  in the new signed permutation are bar-active, as are all spaces which were bar-active in  $\pi$ .

Now suppose we insert  $\bar{n}$  in a bar-active space of  $\pi$  other than the left-most space. Since  $1\bar{3}\bar{2}$  and  $\bar{1}3\bar{2}$  are both forbidden, the only bar-active space left of  $\bar{n}$  in the new signed permutation is the left-most space. But since  $\bar{2}$  does not appear before  $\bar{3}$  in any forbidden pattern, the space immediately right of  $\bar{n}$  is bar-active, as are all spaces to the right of  $\bar{n}$  which were bar-active in  $\pi$ .

Combining the above observations, we find that the children of  $\pi$  have  $3, 4, \dots, f_5(\pi), f_5(\pi)+1, f_5(\pi)+1$  children, and the result follows.  $\square$

## 5 An Isomorphism to a New Schröder Tree

So far we have studied ten sets of forbidden signed permutations, no two of which are trivially Wilf-equivalent, such that if  $T$  is any of these sets then  $|B_n(T)| = r_n$  for all  $n \geq 0$ . Egge has given [4, Eq. (8)] another such set, which is not trivially Wilf-equivalent to any of our sets. All eleven of these sets consist of two patterns of length 2 and two patterns of length 3, so it is natural to seek a list  $T_1, \dots, T_k$  such that the following hold.

- $T_i$  consists of two signed permutations of length 2 and two signed permutations of length 3 for  $1 \leq i \leq k$ .
- $|B_n(T_i)| = r_n$  for all  $n \geq 0$  and all  $i$  such that  $1 \leq i \leq k$ .
- No two of  $T_1, \dots, T_k$  are trivially Wilf-equivalent.
- If  $T$  satisfies the first two conditions above then  $T$  is trivially Wilf-equivalent to  $T_i$  for some  $i$  such that  $1 \leq i \leq k$ .

A computer search suggests that adding the set  $T_{12} = \{21, \bar{2}1, \overline{321}, \overline{312}\}$  to our collection will give us the desired list. It is routine to verify that  $T_{12}$  is not trivially Wilf-equivalent to any of our sets, so in this section we prove that  $|B_n(T_{12})| = r_n$  for all  $n \geq 0$ . To accomplish this, we first introduce a new generating tree.

**Definition 5.1** *The tilted Schröder tree is the generating tree given by*

- *Root:* (2);
- *Rules:* (2)  $\rightarrow$  (2)(4)  
 $(k) \rightarrow (2)(4)(5) \cdots (k)(k+1)(k+1)$  for  $k \geq 4$ .

A preliminary examination of the tilted Schröder tree suggests it has  $r_n$  nodes on level  $n$ . We prove this next.

**Theorem 5.2** *For all  $n \geq 0$ , the tilted Schröder generating tree has exactly  $r_n$  nodes on level  $n$ .*

*Proof.* We use the kernel method. (See [1, 8] for more information on the kernel method.) For any node  $\nu$  in the tilted Schröder tree, let  $level(\nu)$  denote the level of  $\nu$  and let  $label(\nu)$  denote the label of  $\nu$ . Now let  $G(x, y)$  be given by

$$G(x, y) = \sum_{\nu} x^{level(\nu)} y^{label(\nu)},$$

where the sum on the right is over the nodes in the tilted Schröder tree. For all  $k \geq 2$  define  $G_k(x)$  by writing

$$G(x, y) = \sum_{k \geq 2} G_k(x) y^k. \quad (7)$$

We wish to obtain  $G(x, 1)$ . To this end, count the children of each node in the tilted Schröder tree to obtain

$$G(x, y) = y^2 + G_2(x)x(y^2 + y^4) + x \sum_{k \geq 3} G_k(x)(y^2 + y^4 + y^5 + \cdots + y^k + 2y^{k+1}). \quad (8)$$

Since every node has exactly one child with label 2, and the root has label 2, we also have

$$G_2(x) = 1 + xG(x, 1). \quad (9)$$

Use (9) to eliminate  $G_2(x)$  in (8) and use (7) to simplify the result and obtain

$$\begin{aligned} \left(1 - \frac{xy^2}{y-1} - xy\right) G(x, y) &= y^2 + x(y^2 + y^4) + \frac{xy^2 - xy^4}{y-1} \\ &\quad + G(x, 1) \left( x^2(y^2 + y^4) - \frac{x^2y^4 + (1-x)xy^2}{y-1} - x^2y^3 - xy^3(1-x) \right). \end{aligned}$$

Now set  $y = \frac{1+x - \sqrt{1-6x+x^2}}{4x}$  and solve the resulting equation for  $G(x, 1)$  to obtain

$$G(x, 1) = \frac{1-x - \sqrt{x^2 - 6x + 1}}{2x}.$$

Now the result follows from (1).  $\square$

**Remark** Julian West has found an ‘adoption’ procedure which converts the tilted Schröder tree to the classical Schröder tree without changing the number of nodes on any level.

To show that the generating tree for  $B(T_{12})$  is isomorphic to the tilted Schröder tree, we study the bar-active and unbar-active spaces in a signed permutation which avoids  $T_{12}$ . We begin by characterizing the bar-active spaces.

**Lemma 5.3** *Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_{12})$ . Then the following hold.*

- (i) If  $\pi(n-1)$  is unbarred then the right-most space in  $\pi$  is bar-active and all other spaces in  $\pi$  are bar-inactive.
- (ii) If  $\pi(n-1)$  is barred then the right-most two spaces in  $\pi$  are bar-active and all other spaces in  $\pi$  are bar-inactive.

*Proof.* (i) Since no forbidden pattern ends with its largest entry, the right-most space in  $\pi$  is bar-active. However, if we insert  $\bar{n}$  in any space other than the right-most, then  $\bar{n}$  and  $\pi(n-1)$  form a subsequence of type  $\bar{2}1$ , which is forbidden.

(ii) Since no forbidden pattern ends with its largest entry, the right-most space in  $\pi$  is bar-active. Similarly, since no forbidden pattern ends with its largest two entries barred, the second space from the right in  $\pi$  is bar-active. Now suppose we insert  $\bar{n}$  in a space which is not one of the right-most two spaces in  $\pi$ . If  $\pi(n-2)$  is barred then  $\bar{n}$ ,  $\pi(n-2)$ , and  $\pi(n-1)$  form a subsequence of type  $\bar{3}2\bar{1}$  or  $\bar{3}1\bar{2}$ , both of which are forbidden. If  $\pi(n-2)$  is unbarred then  $\bar{n}$  and  $\pi(n-2)$  form a subsequence of type  $\bar{2}1$ , which is also forbidden.  $\square$

Next we characterize the unbar-active spaces.

**Lemma 5.4** Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_{12})$ . Let  $\sigma$  denote the (possibly empty) sequence of barred entries at the right end of  $\pi$ . Then the following hold.

- (i) If  $\sigma$  is empty then the right-most space in  $\pi$  is unbar-active and all other spaces in  $\pi$  are unbar-inactive.
- (ii) Suppose  $\sigma$  is nonempty. Then a space in  $\pi$  is unbar-active if and only if it is adjacent to at least one entry of  $\sigma$ .

*Proof.* (i) Since no forbidden pattern ends with its largest entry, the right-most space in  $\pi$  is unbar-active. However, if we insert  $n$  in any space other than the right-most, then  $n$  and  $\pi(n-1)$  form a subsequence of type  $21$ , which is forbidden.

(ii) This is immediate, since the only forbidden subsequence whose largest entry is unbarred is  $21$ .  $\square$

We conclude the paper by using Lemmas 5.3 and 5.4 to give an isomorphism between the generating tree for  $B(T_{12})$  and the tilted Schröder tree.

**Theorem 5.5** For each signed permutation  $\pi \in B(T_{12})$ , let  $g(\pi) = 2$  if the right-most entry of  $\pi$  is unbarred, and let  $g(\pi)$  denote three plus the number of barred entries at the right end of  $\pi$  otherwise. Then the map  $\pi \mapsto g(\pi)$  is an isomorphism of generating trees between the generating tree for  $B(T_{12})$  and the tilted Schröder generating tree. In particular,

$$|B_n(21, \bar{2}1, \bar{3}2\bar{1}, \bar{3}1\bar{2})| = r_n \quad (n \geq 0).$$

*Proof.* To begin, observe that  $g(\emptyset) = 2$ ,  $g(1) = 2$ , and  $g(\bar{1}) = 4$ , so both trees have the same first two levels, and it is sufficient to show they have the same succession rules.

Fix  $n \geq 2$  and suppose  $\pi \in B_{n-1}(T_{12})$ . We consider two cases.

**Case One:**  $\pi(n-1)$  is unbarred, so that  $g(\pi) = 2$ .

In view of Lemmas 5.3 and 5.4, the signed permutation  $\pi$  has  $2 = g(\pi)$  children, one of which is obtained by inserting  $n$  at the right end of  $\pi$  and the other of which is obtained by inserting  $\bar{n}$  at the right end of  $\pi$ . By the same Lemmas, the child obtained by inserting  $n$  has 2 children and the child obtained by inserting  $\bar{n}$  has 4 children.

**Case Two:**  $\pi(n-1)$  is barred, so that  $g(\pi) \geq 4$ .

In view of Lemmas 5.3 and 5.4, the signed permutation  $\pi$  has  $g(\pi)$  children, two of which are obtained by inserting  $\bar{n}$  and  $g(\pi) - 2$  of which are obtained by inserting  $n$ . By the same Lemmas, both children obtained by inserting  $\bar{n}$  have  $g(\pi) + 1$  children, and the children obtained by inserting  $n$  have  $2, 4, 5, \dots, g(\pi)$  children.

These results correspond with the succession rules for the tilted Schröder generating tree, and the result follows from Theorem 5.2.  $\square$

## References

- [1] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246:29–55, 2002.
- [2] E. Barucci, A. Del Lungo, E. Pergola, and R. Pinzani. Permutations avoiding an increasing number of length-increasing forbidden subsequences. *Discrete Math. Theor. Comput. Sci.*, 4(1):31–44, 2000.
- [3] R. A. Brualdi. *Introductory Combinatorics*. Pearson Prentice Hall, 4th edition, 2004.
- [4] E. S. Egge. Restricted colored permutations and Chebyshev polynomials. preprint.
- [5] S. Gire. *Arbres, permutations à motifs exclus et cartes planaire: quelques problèmes algorithmiques et combinatoires*. PhD thesis, Université Bordeaux I, 1993.
- [6] D. Kremer. Permutations with forbidden subsequences and a generalized Schröder number. *Discrete Math.*, 218:121–130, 2000.
- [7] T. Mansour and J. West. Avoiding 2-letter signed patterns. *Sém. Lothar. Combin.*, 49:Article B49a, 2002.
- [8] H. Prodinger. The kernel method: A collection of examples. *Sém. Lothar. Combin.*, 50:Article B50f, 2003.
- [9] R. Simion. Combinatorial statistics on type-B analogues of noncrossing partitions and restricted permutations. *Electron. J. Combin.*, 7:#R9, 2000.
- [10] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.
- [11] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Math.*, 146:247–262, 1995.

- [12] J. West. Generating trees and forbidden subsequences. *Discrete Math.*, 157:363–374, 1996.