

Connections Between
Diagonal Harmonics
and
Schröder Paths

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Notation

$\lambda \quad \mu$ partitions

$K_{\lambda,\mu}(q, t) :=$ Macdonald's q, t -Kostka polynomial

$$\eta(\mu) := \sum_{i=1}^{l(\mu)} (i-1)\mu_i$$

$$\tilde{K}_{\lambda,\mu}(q, t) := t^{\eta(\mu)} K_{\lambda,\mu}(q, 1/t)$$

$$\tilde{H}_{\mu}[X; q, t] := \sum_{\lambda \vdash n} \tilde{K}_{\lambda,\mu}(q, t) s_{\lambda}$$

∇ is the linear operator with $\nabla \tilde{H}_{\mu} := t^{\eta(\mu)} q^{\eta(\mu')} \tilde{H}_{\mu}$

\langle, \rangle is the bilinear form with $\langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda\mu}$

$$\langle \nabla e_n, f \rangle$$

often has an interpretation in terms of lattice paths

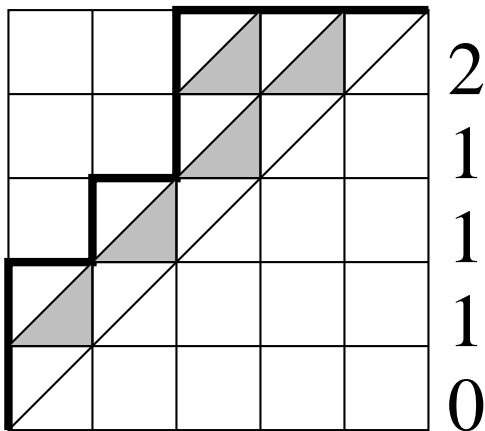
Lattice Paths for $\langle \nabla e_n, s_{1^n} \rangle$

$$\langle \nabla e_n, s_{1^n} \rangle = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}$$

π is a lattice path

“Catalan path”

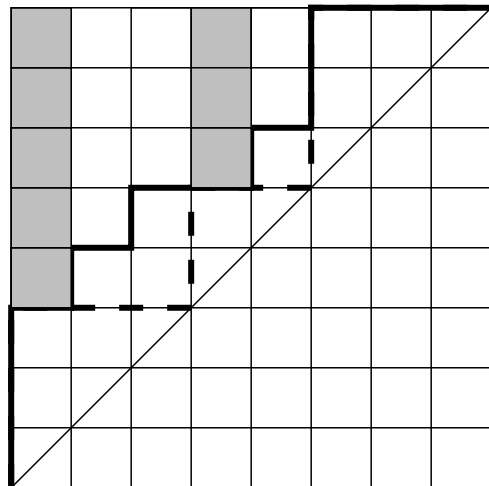
- from $(0, 0)$ to (n, n)
- using only North $(0, 1)$ and East $(1, 0)$ steps
- not passing below $y = x$.



$$\text{area}(\pi) =$$

$$2 + 1 + 1 + 1 + 0$$

$$= 5$$



$$\text{bounce}(\pi) =$$

$$5 + 3 + 0$$

$$= 8$$

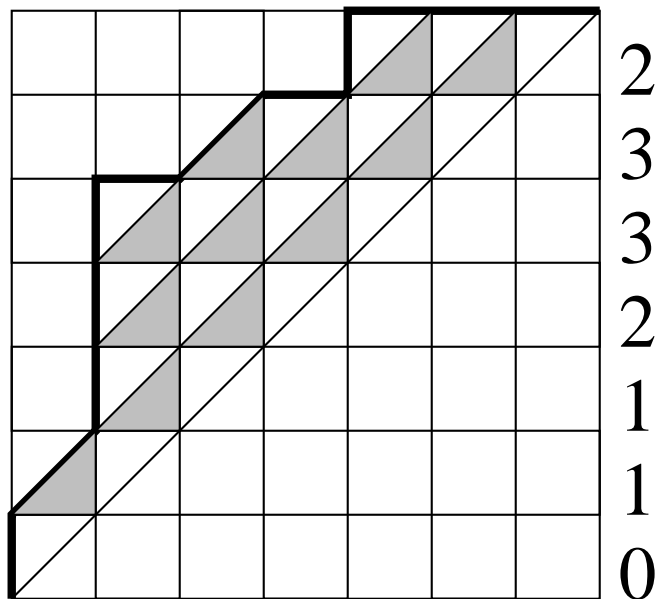
Lattice Paths for $\langle \nabla e_n, e_{n-d} h_d \rangle$

$$\langle \nabla e_n, e_{n-d} h_d \rangle = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}$$

π is a lattice path
with d diagonals

“Schröder path”

- from $(0, 0)$ to (n, n)
- using only North $(0, 1)$, East $(1, 0)$, and Diagonal $(1, 1)$ steps
- not passing below $y = x$.



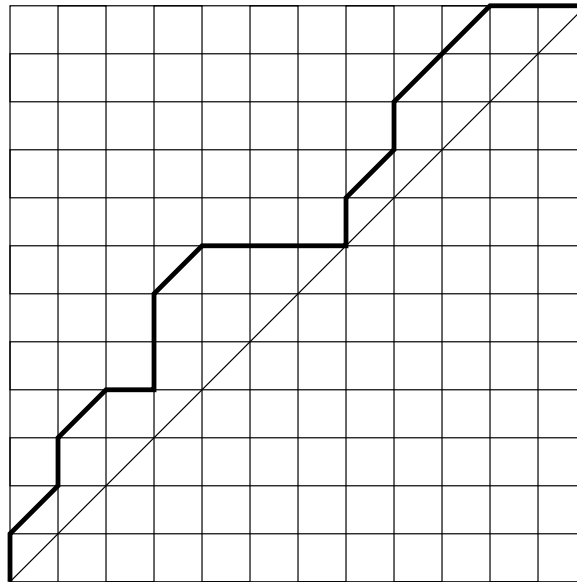
$$\text{area}(\pi) = 2 + 3 + 3 + 2 + 1 + 1 + 0 = 12$$

Lattice Paths for $\langle \nabla e_n, h_j h_k \rangle$

$$\langle \nabla e_n, h_j h_k \rangle = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}$$

π is a Schröder path

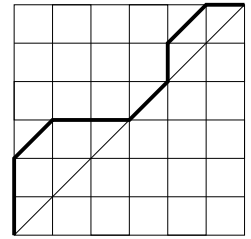
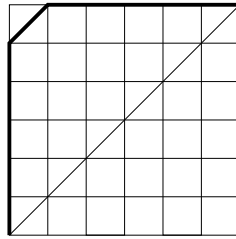
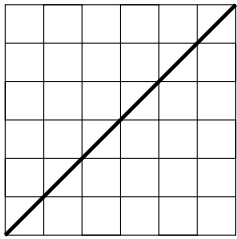
- each NE pair has a D between the N and E
- $j :=$ number of N steps
- $k + 1 :=$ number of D steps



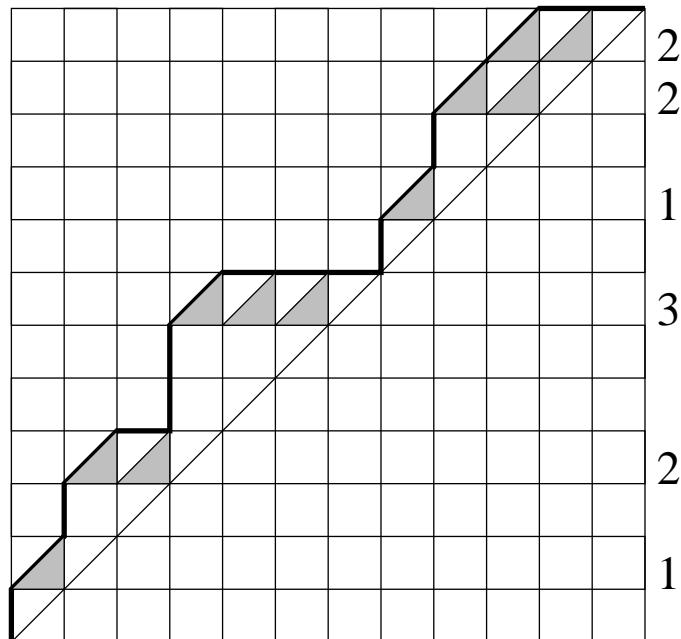
- bounce is usual Schröder bounce

Area for $\langle \nabla e_n, h_j h_k \rangle$

- only rows bounded by a D step contribute
- subtract number of N steps in path



paths with zero area



$$area(\pi) = 2 + 2 + 1 + 3 + 2 + 1 - 6 = 5$$

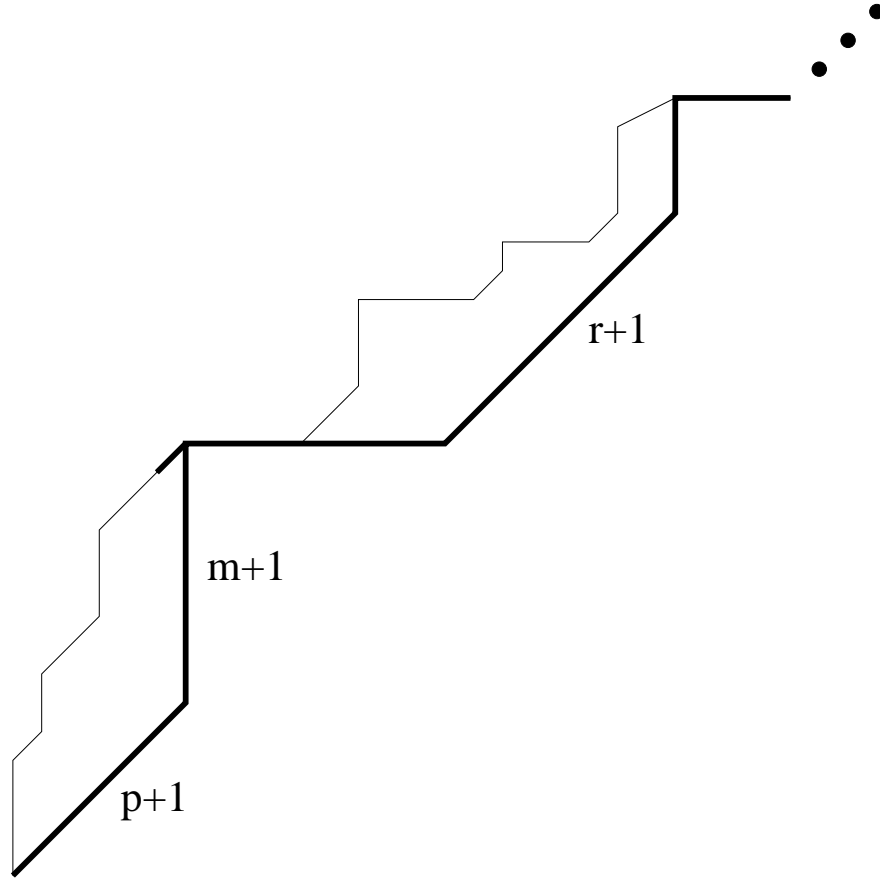
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bijjective proof of symmetry

A Refinement and a Recurrence

$$S_{j,k,p}(q, t) = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}$$

π has exactly $p + 1$ diagonals before first E step



Theorem (EH)

$$S_{j,k,p}(q, t) =$$

$$\sum_{m=0}^{j-1} t^{j-m-1+k-p} \begin{bmatrix} p+m+1 \\ p \end{bmatrix}_q \sum_{r=0}^{k-p-1} \begin{bmatrix} r+m+1 \\ m \end{bmatrix}_q S_{j-m-1, k-p-1, r}$$

A Shorter Recurrence and $t = 1/q$

Theorems (EH)

$$S_{j,k,p}(q, t) = \sum_{m=0}^{j-1} t^{k-p} \begin{bmatrix} p + m + 1 \\ p \end{bmatrix}_q S_{k-p, j-1, m}(q, t)$$

$$S_{j,k,p}(q, 1/q) = q^{j(p-k)} \frac{[p+1]_q}{[k+1]_q} \begin{bmatrix} j+k \\ k \end{bmatrix}_q \begin{bmatrix} j+k-p-1 \\ k-p \end{bmatrix}_q$$

A Catalan Connection

Schröder paths with
diagonal condition



Catalan paths
to (n, n)

$$j + k + 1 = n$$

D



NE

