New Pattern-Avoiding Permutations Counted by the Schröder Numbers

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Theorem

If $\sigma \in S_3$ and $n \geq 0$ then

$$|S_n(\sigma)| = C_n.$$
Theorem

If \( \sigma \in S_3 \) and \( n \geq 0 \) then

\[ |S_n(\sigma)| = C_n. \]

\( C_n = \frac{1}{n+1} \binom{2n}{n} \)

is the number of Catalan paths of length \( n \).

\[ C_n = \sum_{j=1}^{n} C_{j-1} C_{n-j} \]
$r_n$ is the number of paths from $(0, 0)$ to $(n, n)$

- using North $(0, 1)$, East $(1, 0)$ and Diagonal $(1, 1)$ steps and
- never passing below $y = x$. 

A Schröder Path
Schröder Numbers

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- using North \((0, 1)\), East \((1, 0)\) and Diagonal \((1, 1)\) steps and
- never passing below \( y = x \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_n )</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>22</td>
<td>90</td>
<td>394</td>
<td>1806</td>
<td>8558</td>
<td>41586</td>
</tr>
</tbody>
</table>
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- using North $(0, 1)$, East $(1, 0)$ and Diagonal $(1, 1)$ steps and
- never passing below $y = x$.

\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 r_n & 1 & 2 & 6 & 22 & 90 & 394 & 1806 & 8558 & 41586 \\
\end{array}
\]

\[
r_n = r_{n-1} + \sum_{j=1}^{n} r_{j-1}r_{n-j}
\]

\[
r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d}
\]
Observation (E and Mansour)

If \( \pi \in S_n(1243, 2143) \) then at most one entry left of \( n \) is smaller than some entry to the right of \( n \).
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*If \( \pi \in S_n(1243, 2143) \) then at most one entry left of \( n \) is smaller than some entry to the right of \( n \).*

\[
41523 \oplus 34251 = 83967\ 10\ 4251
\]
|S_n(1243, 2143)| = r_{n-1}

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*If \( \pi \in S_n(1243, 2143) \) then at most one entry left of \( n \) is smaller than some entry to the right of \( n \).*

\[
|S_n(, , )| = |S_{n-1}(, , )| + \sum_{j=1}^{n} |S_{j-1}(, , )||S_{n-j}(, , )|
\]
Definition

For \( \pi \in S_n(2413, 3142) \),

- \( L_k > L_{k-1} > \cdots > L_1 \) are the maximal sets of consecutive numbers left of \( n \).
- \( R_1 > R_2 > \cdots > R_l \) are the maximal sets of consecutive numbers right of \( n \).
Stankova’s Proof that $|S_n(2413, 3142)| = r_{n-1}$

**Definition**

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**Observation**

$L_k > R_1 > L_{k-1} > R_2 > \cdots$

or

$R_1 > L_k > R_2 > L_{k-1} > \cdots$
Lemma

- $L_j$ is to the right of $L_{j-1}$ for all $j$ (3142).
- $R_j$ is to the right of $R_{j-1}$ for all $j$ (2413).
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- $R_j$ is to the right of $R_{j-1}$ for all $j$ (2413).
Stankova’s Proof Continues

Lemma

- $L_j$ is to the right of $L_{j-1}$ for all $j$ (3142).
- $R_j$ is to the right of $R_{j-1}$ for all $j$ (2413).

Lemma

If $R_j$ and $L_j$ all avoid 2413 and 3142 then so does $\pi$. 
If $X_n = |S_n(2413, 3142)|$ then

$$X_n = 2 \sum_{\alpha_1 + \cdots + \alpha_m = n-1} X_{\alpha_1} \cdots X_{\alpha_m},$$

which can be rewritten

$$X_n = X_{n-1} + \sum_{j=1}^{n} X_{j-1}X_{n-j}.$$
Are there other sets $R$ such that $|S_n(R)| = r_{n-1}$?
The Question

Are there other sets $R$ such that $|S_n(R)| = r_{n-1}$?

Theorem (Gire, Kremer, West)

If $R \subseteq S_4$ then $|R| = 2$ and $R$ is trivially equivalent to exactly one of

- $1234, 1243$
- $1324, 2314$
- $1342, 2341$
- $3124, 3214$
- $3142, 3214$
- $3412, 3421$
- $1324, 2134$
- $3124, 2314$
- $2134, 3124$
- $2413, 3142$
Avoiding Two Patterns of Length Four


Table 1 (continued)

| Γ         | |S_n(Γ)|, n = 5, 6, 7, 8, 9, 10, 11 | Reference |
|-----------|-------------------------------|-----------|
| 3124,3214 | 90,394,1806,8558,41586,206098,1037718 | Kremer [8] |
| 3142,3214 |                               | West [8]  |
| 3412,3421 |                               |           |
| 1324,2134 |                               |           |
| 3124,2314 |                               |           |
| 2134,3124 |                               |           |
| 2143,2413 | 90,395,1823,8741,43193,218704,1129944 | Open      |
| 1234,1324 | 90,396,1837,8864,44074,224352,1163724 | Zeilbergera |

*aVia the package WILF (see http://www.math.temple.edu/~zeilberger).
Question

Does there exist $\sigma \in S_6$ such that $|S_n(2143, 2413, \sigma)| = r_{n-1}$?
The Conjecture

**Conjecture**

If $\sigma$ is any of the permutations below, then $|S_n(2143, 2413, \sigma)| = r_{n-1}$.

- 415263
- 513642
- 624315
- 465213
- 516324
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If $\sigma$ is any of the permutations below, then $|S_n(2143, 2413, \sigma)| = r_{n-1}$.

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Lemma

If $\pi \in S_n(2143, 2413)$ then $\pi$ has the form below.
The End

Thank You!