Restricted Symmetric Permutations

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Symmetries of Permutations

symmetry group of a square (D_8) acts on S_n

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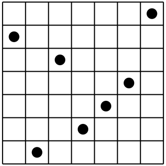

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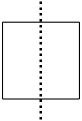

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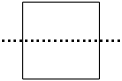

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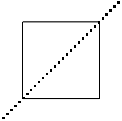

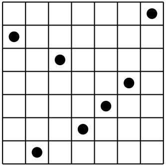

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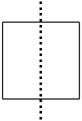

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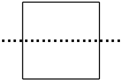

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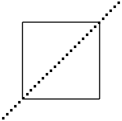

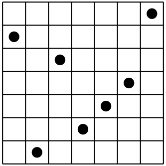

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π contains/avoids σ if and only if π^f contains/avoids σ^f

Subgroups of D_8

- $H_0 = \{e\}$
- $H_1 = \{e, rc\}$
- $H_2 = \{e, i, rc, rci\}$
- $H_3 = \{e, rc, ri, ci\}$
- $H_{4a} = \{e, i\}$
- $H_{4b} = \{e, rci\}$
- $H_5 = \{e, r\}$
- $H_6 = \{e, c\}$
- $H_7 = \{e, r, c, rc\}$
- $H_8 = D_8$

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How many permutations in S_n

- *avoid a set R of patterns and*
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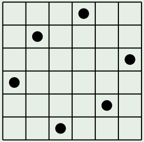
Example



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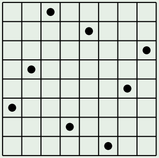

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$|S_{2n}^{rc}(\emptyset)| = |S_{2n+1}^{rc}(\emptyset)| = 2^n n!$

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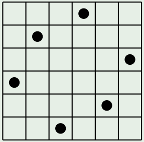

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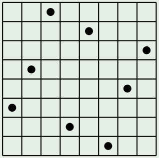

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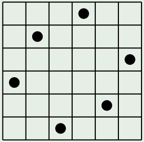

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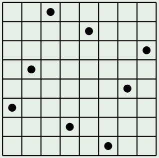

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$$|S_{2n}^{rc}(132, 123)| = |S_{2n+3}^{rc}(132, 123)| = F_{n+1}$$

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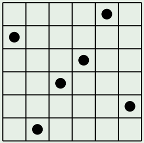

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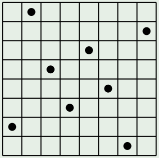

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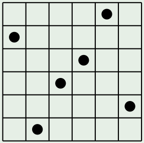

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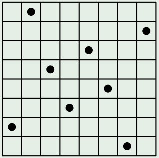

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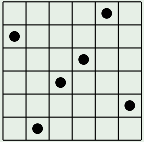

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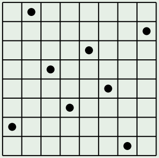

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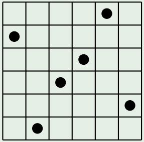

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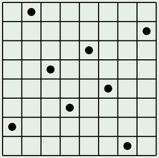

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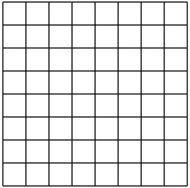

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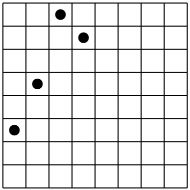

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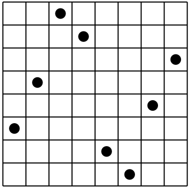

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Example

π avoids $\overline{12}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}, 321$ if and only if π^s avoids 123.

$$|B_n(\overline{12}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}, 321)| = \binom{2n}{n}$$

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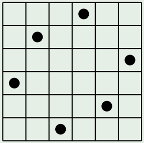

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$$|I_{2n}^{rc}| = 2|I_{2n-2}^{rc}| + (2n-2)|I_{2n-4}^{rc}|$$

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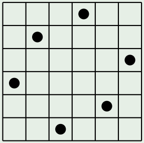

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Theorem (E)

$I_n^{rc}(R) = S_n^{rc}(R)$ if $132 \in R$.

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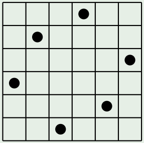

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$$|I_{2n+1}^{rc}(123)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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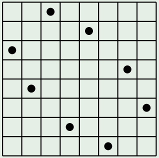

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$$|I_{2n}^{rc}(123)| = 2^n$$

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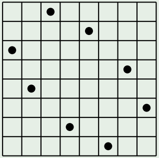

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Theorem (Guibert and Pergola, 2000)

$$|I_{2n}^{rc}(2143)| = \sum_{i=0}^{n} \frac{n!}{(n-i)!\lfloor i/2 \rfloor ! \lceil i/2 \rceil !}.$$

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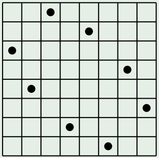

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Example

There are exactly 2^n signed involutions in $B_n(\overline{12}, \overline{1}2, \overline{3}2\overline{1}, \overline{3}21, 32\overline{1}, 321)$.

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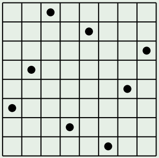

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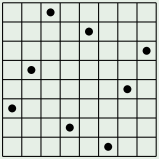

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Theorem (E)

$$|S_{4n}^{90}(1324)| = (n+1)2^{n-1}$$

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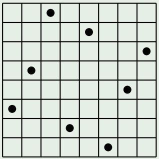

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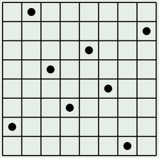

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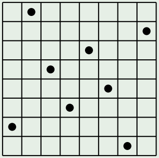

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Example

There are exactly $2^n(2n-1)!!$ signed permutations in B_{2n} with $\overline{\pi}^{-1} = \pi$.

A Conjecture

Theorem

Fix $\sigma \in S_4$.

If $|S_{4n}^{90}(\sigma)|$ is not eventually 0 then σ is Wilf-equivalent to one of 1342, 1324, 2143, 2413.

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If $|S^{90}_{4n}(\sigma)|$ *is not eventually 0 then* σ *is Wilf-equivalent to one of 1342, 1324, 2143, 2413.*

Conjecture

For all $n \geq 0$,

$$|S^{90}_{4n+1}(2413)| = |S^{90}_{4n}(2413)| = d_n,$$

where $d_0 = 1$ *and*

$$d_n = d_{n-1} + \sum_{k=1}^{n} 2^k C_{k-1} d_{n-k}.$$

For Preprints and More Information

www.people.carleton.edu/~eegge