Restricted Symmetric Permutations*

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Abstract

Pattern-avoiding involutions, which have received much enumerative attention, are pattern-avoiding permutations which are invariant under the natural action of a certain subgroup of D_8, the symmetry group of a square. Three other nontrivial subgroups of D_8 also have invariant permutations under this action. For each of these subgroups, we enumerate the set of permutations which are invariant under the action of the subgroup and which also avoid a given set of forbidden patterns. The sets of forbidden patterns we consider include all subsets of S_3. For each subgroup we also give a bijection between the invariant permutations and certain symmetric signed permutations.

Keywords: Pattern-avoiding permutation, pattern-avoiding involution, restricted permutation, signed permutation, signed involution.

1 Introduction and Notation

Let S_n denote the set of permutations of $\{1, \ldots, n\}$, written in one-line notation, and suppose $\pi \in S_n$ and $\sigma \in S_k$. We say a subsequence of π has *type* σ whenever it has all of the same pairwise comparisons as σ, and we say σ is a *subpermutation* of π whenever π has a subsequence of type σ. For example, the subsequence 2869 of the permutation 214538769 has type 1324 so 1324 is a subpermutation of 214538769. We say π *avoids* σ whenever σ is not a subpermutation of π. For example, the permutation 214538769 avoids 312 and 2413, but it has 2586 as a subsequence so it does not avoid 1243. In this context σ is sometimes called a *pattern* or a *forbidden subsequence* and π is sometimes called a *restricted permutation* or a *pattern-avoiding permutation.* For all $n \geq 0$ and any set R of permutations, we write $S_n(R)$ to denote the set of permutations in S_n which avoid every pattern in R.

It is well-known that the symmetry group of a square, which we denote by D_8, acts on permutations in a way that is compatible with pattern avoidance. To describe this action on a given permutation $\pi \in S_n$, we first draw the *diagram of* π, as follows. Begin with an $n \times n$ square, subdivided into n rows of equal height and n columns of equal width, forming n^2 subsquares. Number the rows (resp. columns) of subsquares $1, 2, \ldots, n$ from left to right (resp. bottom to top). To complete the diagram of π, place dots in those squares whose (row,column) coordinates are $(1, \pi(1)), (2, \pi(2)), \ldots, (n, \pi(n))$, where $\pi(i)$ is the ith entry of π. For example, the diagram of 7251643 is pictured below.

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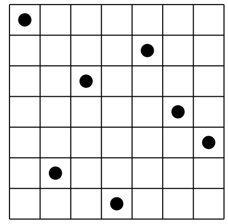


The diagram of 7251643.

The symmetry group of the square acts naturally on these diagrams, and therefore on permutations; for any g in this group we write π^g to denote the image of π under g. We observe that $\pi \in S_n$ contains $\sigma \in S_k$ if and only if there exist k rows and k columns in the diagram of π whose squares of intersection form the diagram of σ. It follows that if g is any symmetry of the square then π avoids σ if and only if π^g avoids σ^g. In this context it is traditional to single out the following three elements of D_8.

- The reverse map r reverses the order of the entries of π. On diagrams r is the reflection over a vertical line.
- The complement map c replaces each entry $\pi(j)$ of $\pi \in S_n$ with $n+1-\pi(j)$. On diagrams c is the reflection over a horizontal line.
- The inverse map i takes a permutation π to its group-theoretic inverse. On diagrams i is the reflection over the diagonal from the lower left corner to the upper right corner.

We note that r, c, and i together generate D_8, each of them has order two, $rc = cr$, $ri = ic$, and $ci = ir$.

It is an easy exercise to show that D_8 has exactly ten subgroups. Here we list these subgroups, using subscripts with the same numerical part for conjugate subgroups and writing e to denote the identity in D_8.

- $H_0 = \{e\}$
- $H_1 = \{e, rc\}$
- $H_2 = \{e, i, rci, rc\}$
- $H_3 = \{e, rc, ri, ci\}$
- $H_{4a} = \{e, i\}$.
- $H_{4b} = \{e, rci\}$
- $H_{5a} = \{e, r\}$
- $H_{5b} = \{e, c\}$
- $H_6 = \{e, r, c, rc\}$
- $H_7 = D_8$.

The permutations which are invariant under H_{4a} (that is, the involutions) and avoid various sets of patterns have been widely studied (a small sample of the literature includes [2, 5, 6, 8, 11]), but pattern-avoiding permutations which are invariant under the other non-identity subgroups of D_8 have received little or no enumerative attention. Indeed, it seems that the only available paper along these lines is [4], in which Guibert and Pergola enumerate 2143-avoiding permutations of even length which are invariant under H_2. In this paper we begin the project of enumerating pattern-avoiding permutations which are invariant under a subgroup of D_8 other than H_{4a} and H_0. No permutation of length 2 or more is invariant under r or c, so no permutation of length 2 or more is invariant under H_{5a}, H_{5b}, H_6, or H_7. In addition, π is invariant under H_{4b} and π avoids σ if and only if π^r is invariant under H_{4a} and π^r avoids σ^r, so enumerating permutations which avoid a given set of patterns and are invariant under H_{4b} is equivalent to enumerating involutions which avoid a given set of patterns. With these observations in mind, we restrict our attention to pattern-avoiding permutations which are invariant under H_1, H_2, or H_3.

For any $n \geq 0$ and any set R of forbidden patterns, we write $S_n^{rc}(R)$ to denote the set of permutations in S_n which are invariant under H_1 (that is, under the reverse-complement map rc) and which avoid every pattern in R; we often abbreviate $S_n^{rc} = S_n^{rc}(\emptyset)$. In Section 2 we find $|S_n^{rc}(R)|$ for various R, including all $R \subseteq S_3$. We begin by using a bijection with signed permutations to show that

$$|S_{2n}^{rc}| = |S_{2n+1}^{rc}| = 2^n n! \qquad (n \geq 0).$$

Next we show that $\pi \in S_n^{rc}(132)$ if and only if π is a layered permutation of a certain form; this characterization allows us to enumerate $S_n^{rc}(R)$ for all $R \subseteq S_3$ with $132 \in R$. In particular, we show that

$$|S_{2n}^{rc}(132)| = |S_{2n+1}^{rc}(132)| = 2^n \qquad (n \geq 0) \tag{1}$$

and

$$|S_{2n}^{rc}(132, 123)| = |S_{2n+3}^{rc}(132, 123)| = F_{n+1} \qquad (n \geq 0), \tag{2}$$

where F_n is the nth Fibonacci number, which is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Next we give a bijection between $S_{2n+1}^{rc}(123)$ and $S_n(123)$ to show that

$$|S_{2n+1}^{rc}(123)| = C_n \qquad (n \geq 0),$$

where C_n is the nth Catalan number, which is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. Finally, we use generating functions and the kernel method to show that

$$|S_{2n}^{rc}(123)| = \binom{2n}{n} \qquad (n \geq 0).$$

For any $n \geq 0$ and any set R of forbidden patterns, we write $I_n^{rc}(R)$ to denote the set of permutations in S_n which are invariant under H_2 and which avoid every pattern in R. Observe that these are the involutions in $S_n^{rc}(R)$. We often abbreviate $I_n^{rc} = I_n^{rc}(\emptyset)$. In Section 3 we find $|I_n^{rc}(R)|$ for various R, including all $R \subseteq S_3$. We begin by showing that $|I_{2n+1}^{rc}| = |I_{2n}^{rc}|$ for all $n \geq 0$ and that our bijection between S_{2n}^{rc} and the set of signed permutations of length n gives rise to a bijection between I_{2n}^{rc} and the set of signed involutions of length n. This fact implies that

$$|I_{2n}^{rc}| = 2|I_{2n-2}^{rc}| + (2n-2)|I_{2n-4}^{rc}| \qquad (n \geq 2).$$

Next we show that $I_n^{rc}(132) = S_n^{rc}(132)$ for all $n \geq 0$, which allows us to use results like (1) and (2) to enumerate $I_n^{rc}(R)$ for all $R \subseteq S_3$ with $132 \in R$. To conclude Section 3 we first use a result of Simion and Schmidt [11] to show that

$$|I_{2n+1}^{rc}(123)| = \binom{n}{\lfloor \frac{n}{2} \rfloor} \qquad (n \geq 0). \tag{3}$$

We then use elementary methods to show that

$$|I_{2n}^{rc}(123)| = 2^n \qquad (n \geq 0). \tag{4}$$

Finally, for any $n \geq 0$ and any set R of forbidden patterns, we write $S_n^{90}(R)$ to denote the set of permutations in S_n which are invariant under H_3 and which avoid every pattern in R; we often abbreviate $S_n^{90} = S_n^{90}(\emptyset)$. In Section 4 we find $|S_n^{90}(R)|$ for various R, including all $R \subseteq S_3$ and all but one R consisting of a single pattern of length 4. We begin by showing that $|S_{4n+2}^{90}| = |S_{4n+3}^{90}| = 0$ for all $n \geq 0$ and

$$|S_{4n}^{90}| = |S_{4n+1}^{90}| = 2^n(2n-1)(2n-3)\cdots 3 \cdot 1 = 2^n(2n-1)!! \qquad (n \geq 0). \tag{5}$$

Next we use elementary methods to show that if $\sigma \in S_3$ then $|S_n^{90}(\sigma)| = 0$ for all $n \geq 5$, which implies that $|S_n^{90}(R)| = 0$ for all $n \geq 5$ and all R which contain a pattern of length 3. Turning our attention to forbidden subsequences of length 4, we show that if $n \geq 5$ then $|S_n^{90}(1243)| = |S_n^{90}(1432)| = 0$, and that $|S_{4n}^{90}(1342)| = 2$ and $|S_{4n+1}^{90}(1342)| = 1$ for all $n \geq 1$. We conclude by showing that

$$|S_{4n}^{90}(1324)| = (n+1)2^{n-1} \qquad (n \geq 1), \tag{6}$$

$$|S_{4n+1}^{90}(1324)| = 2^n \qquad (n \geq 0), \tag{7}$$

and

$$|S_{4n+1}^{90}(2143)| = |S_{4n}^{90}(2143)| = \binom{2n}{n} \qquad (n \geq 1). \tag{8}$$

We prove (6)–(8) by enumerating the permutations in question according to the position of their largest element.

2 Permutations Invariant Under the Reverse Complement

Recall from the Introduction that for any $n \geq 0$ and any set R of forbidden patterns we write $S_n^{rc}(R)$ to denote the set of permutations in S_n which are invariant under H_1 (that is, under the reverse-complement map rc) and which avoid every pattern in R. Our goal in this section is to enumerate $S_n^{rc}(R)$ for various R. Before we begin these enumerations, we recall the signed permutations and describe a useful connection they have with $S_n^{rc}(R)$.

A *signed permutation* of length n is a permutation of $\{1, 2, \ldots, n\}$, written in one-line notation, in which each entry may or may not have a bar over it. We write B_n to denote the set of signed permutations of length n, and we identify S_n with the set of signed permutations in B_n in which no entry is barred. As we show next, there is a natural bijection between B_n and S_{2n}^{rc}.

Definition 2.1 *Fix $n \geq 0$. For all $\pi \in B_n$ we write π^s to denote the permutation in S_{2n} given by*

$$\pi^s(j) = \begin{cases} \pi(j) & \text{if } \pi(j) \text{ is barred and } 1 \leq j \leq n \\ 2n+1-\pi(j) & \text{if } \pi(j) \text{ is unbarred and } 1 \leq j \leq n \\ 2n+1-\pi(2n+1-j) & \text{if } \pi(2n+1-j) \text{ is barred and } n+1 \leq j \leq 2n \\ \pi(2n+1-j) & \text{if } \pi(2n+1-j) \text{ is unbarred and } n+1 \leq j \leq 2n \end{cases}$$

for all j with $1 \leq j \leq 2n$. Here we write $\pi(j)$ to denote the jth (numerical) entry of π. For instance, if $\pi = 31\overline{2}$ then $\pi(1) = 3$ and $\pi(3) = 2$, while $\pi^s = 462513$.

Proposition 2.2 *For all $n \geq 0$ the map $\pi \mapsto \pi^s$ is a bijection between B_n and S_{2n}^{rc}.*

Proof. First note that if $\pi \in B_n$ then $\pi^s \in S_{2n}^{rc}$ by construction. Now observe that if $\sigma \in S_{2n}^{rc}$ and we define $\pi \in B_n$ by setting

$$\pi(j) = \begin{cases} \overline{\sigma(j)} & \text{if } 1 \leq \sigma(j) \leq n \\ 2n+1-\sigma(j) & \text{if } n+1 \leq \sigma(j) \leq 2n \end{cases}$$

for all j with $1 \leq j \leq n$ then $\pi^s = \sigma$. It follows that s is a bijection, as desired. □

It will be useful later to have a certain alternative description of the map $\pi \mapsto \pi^s$. To obtain this description, first note that we may view any $\pi \in B_n$ as a bijection from $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ to itself if we set $\pi(\overline{j}) = \overline{\pi(j)}$, where $\overline{\overline{j}} = j$ for all j with $1 \leq j \leq n$. For example, if $\pi = 3\overline{4}12$ then the resulting bijection f is given by

i	1	2	3	4	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$
$f(i)$	3	$\overline{4}$	1	2	$\overline{3}$	4	$\overline{1}$	$\overline{2}$

Now replace $1, 2, \ldots, n$ with $2n, 2n-1, \ldots, n+1$, respectively, and then replace $\overline{1}, \overline{2}, \ldots, \overline{n}$ with $1, 2, \ldots, n$, respectively. When $\pi = 3\overline{4}12$ the resulting bijection g is given by

i	8	7	6	5	1	2	3	4
$g(i)$	6	4	8	7	3	5	1	2

To obtain π^s, view this bijection as a permutation in S_{2n} and take its reverse. When $\pi = 3\overline{4}12$ we obtain $\pi^s = 64872153$.

Having briefly considered S_n^{rc} when n is even, we now turn our attention to S_n^{rc} when n is odd. The following map will be useful in this case.

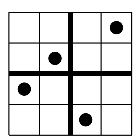
Definition 2.3 *Fix $n \geq 0$. For all $\pi \in S_{2n}$ we write π^u to denote the permutation in S_{2n+1} given by*

$$\pi^u(j) = \begin{cases} \pi(j) & \text{if } 1 \leq j \leq n \text{ and } \pi(j) < n+1 \\ \pi(j)+1 & \text{if } 1 \leq j \leq n \text{ and } \pi(j) \geq n+1 \\ n+1 & \text{if } j = n+1 \\ \pi(j-1) & \text{if } n+2 \leq j \leq 2n+1 \text{ and } \pi(j) < n+1 \\ \pi(j-1)+1 & \text{if } n+2 \leq j \leq 2n+1 \text{ and } \pi(j) \geq n+1 \end{cases}$$

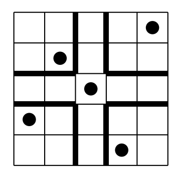
for all j with $1 \leq j \leq 2n+1$.

As the example below illustrates, u acts on a given permutation π by adding a dot in the center of the diagram of π.

Example 2.4 *If $\pi = 2314$ then $\pi^u = 24315$.*



The diagram of $\pi = 2314$.



The diagram of $\pi^u = 24315$.

With the maps s and u in hand, we are ready to begin our enumerations of $S_n^{rc}(R)$. We first consider S_n^{rc} itself.

Theorem 2.5 *For all $n \geq 0$ we have*

$$|S_{2n}^{rc}| = |S_{2n+1}^{rc}| = 2^n n!. \tag{9}$$

Proof. It is well-known that $|B_n| = 2^n n!$ for all $n \geq 0$, so $|S_{2n}^{rc}| = 2^n n!$ for all $n \geq 0$ by Proposition 2.2. Moreover, if $\pi \in S_{2n+1}^{rc}$ then $\pi(n+1) = 2n+2-\pi(2n+2-(n+1))$ so $\pi(n+1) = n+1$, and it is routine to show that $\pi \in S_{2n}^{rc}$ if and only if $\pi^u \in S_{2n+1}^{rc}$. Therefore u is a bijection between S_{2n}^{rc} and S_{2n+1}^{rc} and (9) follows. □

Our goal for the rest of this section is to enumerate $S_n^{rc}(R)$ for various sets of forbidden patterns. The symmetry group D_8 allows us to significantly reduce the number of sets R we must consider.

Lemma 2.6 *For any $g \in D_8$ and any set R of patterns, the following hold for all $n \geq 0$.*

(i) *$\pi \in S_n^{rc}(R)$ if and only if $\pi^g \in S_n^{rc}(R^g)$. In particular, $|S_n^{rc}(R)| = |S_n^{rc}(R^g)|$.*

(ii) *$S_n^{rc}(R) = S_n^{rc}(R^{rc}) = S_n^{rc}(R \cup R^{rc})$.*

Here $R^h = \{\sigma^h \mid \sigma \in R\}$ for all $h \in D_8$.

Proof. (i) Fix $g \in D_8$ and any set R of patterns, and recall that π avoids R if and only if π^g avoids R^g. Moreover, rc is in the center of D_8 so $\pi^{rcg} = \pi^{grc}$. Therefore π is invariant under rc if and only if π^g is invariant under rc.

(ii) Suppose $\pi \in S_n$ is invariant under rc. Then π avoids R if and only if $\pi^{rc} = \pi$ avoids R^{rc}. In other words, $\pi \in S_n^{rc}(R)$ if and only if $\pi \in S_n^{rc}(R^{rc})$, so $S_n^{rc}(R) = S_n^{rc}(R^{rc})$. Using this we now have $S_n^{rc}(R \cup R^{rc}) = S_n^{rc}(R) \cap S_n^{rc}(R^{rc}) = S_n^{rc}(R)$, as desired. □

By Lemma 2.6(i), the elements of D_8 provide, for all $n \geq 0$, a bijection from $S_n^{rc}(123)$ to $S_n^{rc}(321)$ and bijections among $S_n^{rc}(132)$, $S_n^{rc}(213)$, $S_n^{rc}(231)$, and $S_n^{rc}(312)$. Consequently, to enumerate $S_n^{rc}(\sigma)$ for $\sigma \in S_3$ we need only consider $\sigma = 132$ and $\sigma = 123$. We begin with $\sigma = 132$, showing that elements of $S_n^{rc}(132)$ have a special form.

Definition 2.7 *For all positive integers $l_1, \ldots, l_k$ we write $\langle l_1, \ldots, l_k \rangle$ to denote the layered permutation consisting of k consecutive increasing sequences, with lengths $l_1, l_2, \ldots, l_k$. In particular, $\langle l_1, \ldots, l_k \rangle$ is given by*

$$n - l_1 + 1, n - l_1 + 2, \ldots, n, n - l_1 - l_2 + 1, \ldots, n - l_1 - 1, n - l_1, \ldots, 1, 2, \ldots, l_k,$$

where $n = l_1 + \cdots + l_k$. For instance, $\langle 3, 1, 1, 2 \rangle = 5674312$.

Lemma 2.8 *The following are equivalent for all $\pi \in S_{2n}$.*

(i) $\pi \in S_{2n}^{rc}(132)$.

(ii) *There exist positive integers $l_1, \ldots, l_k$ such that $\pi = \langle l_1, \ldots, l_k, l_k, \ldots, l_1 \rangle$ or there exist positive integers $l_1, \ldots, l_k$ such that $\pi = \langle l_1, \ldots, l_{k-1}, 2l_k, l_{k-1}, \ldots, l_1 \rangle$.*

Proof. By Lemma 2.6(ii) we have $S_n^{rc}(132) = S_n^{rc}(132, 213)$ for all $n \geq 0$. Simion and Schmidt have shown [11, Prop. 8] that π avoids 132 and 213 if and only if π is layered in the sense of Definition 2.7. Moreover, it is routine to show that $\langle l_1, \ldots, l_k \rangle^{rc} = \langle l_k, \ldots, l_1 \rangle$ for all $l_1, \ldots, l_k$, and the result follows. □

Lemma 2.9 *The following are equivalent for all $\pi \in S_{2n+1}$.*

(i) $\pi \in S_{2n+1}^{rc}(132)$.

(ii) *There exist positive integers $l_1, \ldots, l_k$ such that $\pi = \langle l_1, \ldots, l_{k-1}, 2l_k - 1, l_{k-1}, \ldots, l_1 \rangle$.*

Proof. This is similar to the proof of Lemma 2.8. □

Knowing the form of the elements of $S_n^{rc}(132)$ allows us to obtain the following enumerations.

Theorem 2.10 *We have*

(i)

$$|S_{2n}^{rc}(132)| = |S_{2n}^{rc}(132, 213)| = 2^n \qquad (n \geq 0) \tag{10}$$

and

$$|S_{2n+1}^{rc}(132)| = |S_{2n+1}^{rc}(132, 213)| = 2^n \qquad (n \geq 0); \tag{11}$$

(ii)

$$|S_n^{rc}(132, 231)| = 2 \qquad (n \geq 2) \tag{12}$$

and

$$|S_n^{rc}(132, 123, 231)| = |S_n^{rc}(132, 321, 231)| = 1 \qquad (n \geq 3); \tag{13}$$

(iii)

$$|S^{rc}_{2n}(132,123)| = F_{n+1} \qquad (n \geq 0) \tag{14}$$

and

$$|S^{rc}_{2n+1}(132,123)| = F_{n} \qquad (n \geq 0); \tag{15}$$

(iv)

$$|S^{rc}_{2n}(132,321)| = 2 \qquad (n \geq 1) \tag{16}$$

and

$$|S^{rc}_{2n+1}(132,321)| = 1 \qquad (n \geq 0). \tag{17}$$

Proof. (i) By Lemma 2.8 we know that $\pi \in S^{rc}_{2n}(132)$ if and only if $\pi = \langle l_1, \ldots, l_k, l_k, \ldots, l_1\rangle$ or $\pi = \langle l_1, \ldots, l_{k-1}, 2l_k, l_{k-1}, \ldots, l_1\rangle$, where $l_1 + \cdots + l_k = n$. Permutations of the first form are in bijection with compositions of n, as are permutations of the second form, so there are 2^{n-1} permutations of each form. Combining these yields (10).

The proof of (11) is similar to the proof of (10) using Lemma 2.9.

(ii) By Lemma 2.6(ii) we have $S^{rc}_n(132,231) = S^{rc}_n(132,213,231,312)$ for all $n \geq 0$, and it follows that $S^{rc}_n(132,231) = \{12\ldots n, n\ldots 21\}$ for all $n \geq 2$.

(iii) By Lemma 2.8 we know that $\pi \in S^{rc}_{2n}(132,123)$ if and only if $\pi = \langle l_1, \ldots, l_k, l_k, \ldots, l_1\rangle$ or $\pi = \langle l_1, \ldots, l_{k-1}, 2l_k, l_{k-1}, \ldots, l_1\rangle$, where $l_1 + \cdots + l_k = n$ and each layer has length at most 2. Permutations of the first form are in bijection with compositions of n using only 1s and 2s, of which there are known to be F_n. Permutations of the second form are in bijection with compositions of $n-1$ using only 1s and 2s, of which there are known to be F_{n-1}. Combining these yields (14).

The proof of (15) is similar to the proof of (14).

(iv) This is similar to the proof of (iii). □

Suppose R_1 and R_2 are sets of permutations. We say R_1 and R_2 are H_1-*Wilf equivalent* whenever there exists N such that $|S^{rc}_n(R_1)| = |S^{rc}_n(R_2)|$ for all $n \geq N$. By Lemma 2.6, most subsets of S_3 are H_1-Wilf equivalent to a set of forbidden patterns in Theorem 2.10. The exceptions are $\{123\}$ and those sets which contain $\{123, 321\}$. If a set R is of the latter type then then by the Erdös-Szekeres Theorem we have $|S_n(R)| = 0$ for all $n \geq 5$. With these observations in mind, we turn our attention to $S^{rc}_n(123)$, beginning with the case in which n is odd. To handle this case, we first note that the map su is a bijection from B_n to S^{rc}_{2n+1} for all $n \geq 0$. Our main tool is the following characterization of those $\pi \in B_n$ which map to $S^{rc}_{2n+1}(123)$ under this bijection.

Lemma 2.11 *Fix $n \geq 0$. Then $\pi^{su} \in S^{rc}_{2n+1}(123)$ if and only if $\pi \in S_n(321)$.*

Proof. ($\Longrightarrow$) Suppose $\pi^{su} \in S^{rc}_{2n+1}(123)$. If $\pi(j)$ is barred then $\pi^{su}(j)\pi^{su}(n+1)\pi^{su}(2n+2-j)$ has type 123, so $\pi \in S_n$. Now if $\pi(j_1)\pi(j_2)\pi(j_3)$ has type 321 then $\pi^{su}(j_1)\pi^{su}(j_2)\pi^{su}(j_3)$ has type 123, so π avoids 321. Therefore $\pi \in S_n(321)$.

($\Longleftarrow$) Suppose $\pi \in S_n(321)$ and π^{su} contains a subsequence σ of type 123. Since $\pi \in S_n$, every entry of π^{su} to the left of $\pi^{su}(n+1) = n+1$ is greater than every entry to the right of $n+1$, so σ is entirely to the left of $n+1$ or entirely to the right of $n+1$. If necessary, replace σ with its image under rc so that σ is to the left of $n+1$. Then σ corresponds to a subsequence of π of type 321, a contradiction. Therefore $\pi^{su} \in S^{rc}_{2n+1}(123)$. □

Lemma 2.11 leads directly to an enumeration of $S^{rc}_{2n+1}(123)$.

Theorem 2.12 *For all $n \geq 0$ we have*

$$|S^{rc}_{2n+1}(123)| = C_n, \tag{18}$$

where C_n is the nth Catalan number.

Proof. By Lemma 2.11 we have $|S^{rc}_{2n+1}(123)| = |S_n(321)|$ for all $n \geq 0$. But it is well-known that $|S_n(321)| = C_n$, and (18) follows. □

Lemma 2.11 can also be used to enumerate $S^{rc}_{2n+1}(R)$ for other R which contain 123. As an example, we next enumerate $S^{rc}_{2n+1}(123, 2143)$.

Theorem 2.13 *For all $n \geq 1$ we have*

$$|S^{rc}_{2n+1}(123, 2143)| = F_{2n-2}. \tag{19}$$

Proof. By Lemma 2.11 we have $\pi^{su} \in S^{rc}_{2n+1}(123)$ if and only if $\pi \in S_n(321)$. Now suppose $\pi^{su} \in S^{rc}_{2n+1}(123)$ contains a subsequence σ of type 2143. Since the first entry of 2143 is less than the last entry, σ must be entirely to the left of $n+1$ or entirely to the right of $n+1$. Since $2143 = 2143^{rc}$, we may assume σ is entirely to the left of $n+1$, so π contains a subsequence of type $2143^c = 3412$. Therefore π^{su} avoids 2143 if and only if π avoids 3412 and the map su is a bijection between $S^{rc}_{2n+1}(123, 2143)$ and $S_n(321, 3412)$. Now the result follows from [12, Ex. 9]. □

We now turn our attention to $|S^{rc}_n(123)|$ when n is even. To enumerate these permutations we need some preliminary results concerning their structure. In particular, it is clear that if π is in $S^{rc}_{2n}(123)$ and we remove 1 and $2n$ from π and decrease each remaining entry by 1 then the resulting permutation π' is in $S^{rc}_{2n-2}(123)$. In this case we say π is a *child* of π', or π' is a *parent* of π. However, inserting 1 and $2n$ into a permutation in $S^{rc}_{2n-2}(123)$ does not always produce a permutation in $S^{rc}_{2n}(123)$. In the next three lemmas we characterize those insertions into a parent permutation which produce children in the correct class.

Lemma 2.14 *Suppose $\pi \in S^{rc}_{2n}(123)$ and 1 appears before $2n$ in π. Then $\pi(n) = 1$ and $\pi(n+1) = 2n$.*

Proof. Since $(2n+1) - 1 = 2n$ and $\pi^{rc} = \pi$, one of 1 and $2n$ is among $\pi(1), \ldots, \pi(n)$ and the other is among $\pi(n+1), \ldots, \pi(2n)$. Since 1 appears before $2n$ in π, it must be that 1 is among $\pi(1), \ldots, \pi(n)$ and $2n$ is among $\pi(n+1), \ldots, \pi(2n)$. But if $\pi(n) \neq 1$ then 1 $\pi(n)$ $2n$ is a subsequence of type 123, contradicting our assumption that π avoids 123. Thus $\pi(n) = 1$ and $\pi(n+1) = 2n + 1 - \pi(2n+1-n-1) = 2n$. □

In the next lemma we count those permutations in $S^{rc}_{2n-2}(123)$ with a child in $S^{rc}_{2n}(123)$ in which 1 appears before $2n$.

Lemma 2.15 *Fix $n \geq 1$. For each $\pi \in S^{rc}_{2n-2}(123)$ let π' denote the permutation in S_{2n} defined by*

$$\pi'(j) = \begin{cases} \pi(j) + 1 & \text{if } j < n \\ 1 & \text{if } j = n \\ 2n & \text{if } j = n+1 \\ \pi(j-2) + 1 & \text{if } j > n+1. \end{cases}$$

Then the following hold.

(i) $\pi' \in S^{rc}_{2n}$.

(ii) *π' avoids 123 if and only if $\pi(j) > \pi(j+1)$ for all j with $1 \leq j \leq n-2$.*

(iii) *There are exactly 2^{n-1} permutations in $S^{rc}_{2n-2}(123)$ for which $\pi' \in S^{rc}_{2n}(123)$.*

We call a permutation $\pi \in S^{rc}_{2n-2}(123)$ a brahmin *whenever $\pi' \in S^{rc}_{2n}(123)$.*

Proof. (i) This is immediate from the definition of π' and the fact that $\pi^{rc} = \pi$.

(ii) ($\Longrightarrow$) Suppose there exists j with $1 \leq j \leq n-2$ such that $\pi(j) < \pi(j+1)$. Then $\pi'(j)\pi'(j+1)\pi'(n+1)$ is a subsequence of type 123 in π'.

($\Longleftarrow$) Suppose π' contains a subsequence σ of type 123. Since π contains no such subsequence, at least one of 1 and $2n$ is an entry in σ. But 1 and $2n$ appear in decreasing order in π' so they are not both entries in σ. Replacing σ with its image under rc if necessary, we may assume $2n$ is an entry in σ. Since $2n$ must play the role of the 3 in σ, the entries $\pi(1), \ldots, \pi(n-1)$ are not all decreasing, so there exists j with $1 \leq j \leq n-2$ such that $\pi(j) < \pi(j+1)$.

(iii) Suppose $\pi \in S^{rc}_{2n-2}(123)$ is a brahmin, and consider the pairs $\{j, 2n-1-j\}$ for $1 \leq j \leq n-1$. Since $\pi^{rc} = \pi$, exactly one element of each of these pairs appears among $\pi(1), \ldots, \pi(n-1)$, so there are 2^{n-1} ways to choose the set $\{\pi(1), \ldots, \pi(n-1)\}$. Now (ii) determines $\pi(1), \ldots, \pi(n-1)$ and the fact that $\pi^{rc} = \pi$ determines the rest of π. It is immediate that all permutations constructed in this way are brahmins. □

Next we describe how to obtain permutations in $S^{rc}_{2n}(123)$ in which $2n$ appears before 1 as children of permutations in $S^{rc}_{2n-2}(123)$.

Lemma 2.16 *Fix $n \geq 1$ and j with $1 \leq j \leq n+1$. For each permutation $\pi \in S^{rc}_{2n-2}(123)$, let π' denote the permutation in S_{2n} obtained by increasing each entry of π by 1, inserting $2n$ immediately after the $(j-1)$th entry of π, and inserting 1 immediately before the $(2n-j)$th entry of π. Then the following hold.*

(i) *$\pi' \in S^{rc}_{2n}$.*

(ii) *π' avoids 123 if and only if $\pi(k) > \pi(k+1)$ for all $k < j$.*

Proof. (i) This is immediate from the definition of π' and the fact that $\pi^{rc} = \pi$.

(ii) This is similar to the proof of Lemma 2.15(ii). □

We are now ready to enumerate $S^{rc}_{2n}(123)$.

Theorem 2.17 *For all $n \geq 0$ we have*

$$|S^{rc}_{2n}(123)| = \binom{2n}{n}. \tag{20}$$

Proof. For each $\pi \in S^{rc}_{2n}(123)$ set $lth(\pi) = n$ and let $child(\pi)$ denote the number of children π has in $S^{rc}_{2n+2}(123)$. Observe that $child(\pi) \geq 2$ for all $\pi \in S^{rc}_{2n}(123)$ by Lemma 2.16(ii). With this notation, let $F(x, y)$ denote the generating function

$$F(x, y) = \sum x^{lth(\pi)} y^{child(\pi)},$$

where the sum is over all permutations in $S^{rc}_{2n}(123)$ which are not brahmins. Moreover, for all $k \geq 2$, define $F_k(x)$ by writing

$$F(x, y) = \sum_{k=2}^{\infty} F_k(x) y^k.$$

Note that in view of Lemma 2.15(iii) the coefficient of x^n in $F(x, 1) + \frac{1}{1-2x}$ is $|S^{rc}_{2n}(123)|$. We use the kernel method to obtain this generating function. (See [1, 9] for more information on the kernel method.)

To begin, note that since $F(x, y)$ is a sum over non-brahmins, by Lemma 2.15(ii) every permutation which contributes to $F(x, y)$ contains a decrease, and is therefore the child of either a brahmin or a non-brahmin of (ordinary) length at least four. By Lemmas 2.14, 2.15(ii), and 2.16(ii), each brahmin in S_{2n} has two children who are brahmins and n children who are not brahmins. By Lemma 2.16(ii) these non-brahmin children have $2, 3, \ldots, n+1$ children, all of whom are non-brahmins. Similarly, by Lemma 2.16(ii), if a non-brahmin in $S_{2n}^{rc}(123)$ has k children then none of those children are brahmins, and they have $k+1, 2, 3, \ldots, k$ children, all of whom are also non-brahmins. Putting these observations together, we find

$$F(x, y) = \sum_{n=1}^{\infty} 2^n x^{n+1} (y^2 + y^3 + \cdots + y^{n+1}) + x \sum_{k=2}^{\infty} F_k(x)(y^{k+1} + y^2 + y^3 + \cdots + y^k), \tag{21}$$

where the first sum accounts for the children of the brahmins and the second accounts for the children of the non-brahmins. From (21) it follows that

$$F(x, y) = \frac{xy^2}{y-1} \left(\frac{2xy}{1-2xy} - \frac{2x}{1-2x} \right) + \frac{xy^2}{y-1} \left(F(x, y) - F(x, 1) \right). \tag{22}$$

Now set $y = \frac{1-\sqrt{1-4x}}{2x}$ and use the fact that $\frac{xy^2}{y-1} = 1$ to simplify (22) and obtain

$$F(x, 1) = \frac{2xy}{1-2xy} - \frac{2x}{1-2x}. \tag{23}$$

Add $\frac{1}{1-2x}$ to both sides of (23), replace y with $\frac{1-\sqrt{1-4x}}{2x}$, and simplify the result to obtain

$$F(x, 1) + \frac{1}{1-2x} = \frac{1}{\sqrt{1-4x}}.$$

It is well-known that the coefficient of x^n in $\frac{1}{\sqrt{1-4x}}$ is $\binom{2n}{n}$, and (20) follows. □

Theorem 2.17 completes our analysis of the H_1-Wilf-equivalence classes of subsets of S_3. In the table below we summarize the H_1-Wilf-equivalence classes for which $|S_n^{rc}(R)| \neq 0$.

$\|R\|$	R	$\|S_n^{rc}(R)\|$
0	$\emptyset$	$2^n n!$
1	$\{123\}$	Equations (18) and (20)
	$\{132\}$	Equations (10) and (11)
2	$\{132, 213\}$	Equations (10) and (11)
	$\{132, 231\}$	Equation (12)
	$\{132, 123\}$	Equations (14) and (15)
	$\{132, 321\}$	Equations (16) and (17)
3	$\{132, 123, 231\}$	Equation (13)

H_1-Wilf-equivalence classes for $R \subseteq S_3$

We observed in Proposition 2.2 that S_{2n}^{rc} is closely connected with signed permutations of length n. In our next result, we extend this connection to include pattern avoidance. To describe how to extend the notion of pattern avoidance to signed permutations, first suppose $\pi \in B_n$ and $\sigma \in B_k$. We say a subsequence of π has *type* σ whenever it has all of the same pairwise comparisons as σ and an entry in the subsequence of π is barred if and only if the corresponding entry in σ is barred. We say σ is a *subpermutation* of π whenever π has a subsequence of type σ. For example, the subsequence $3\bar{4}71$ of the signed permutation $93\bar{4}72\bar{8}\bar{5}16$ has type $2\bar{3}41$, so $2\bar{3}41$ is a subpermutation of $93\bar{4}72\bar{8}\bar{5}16$. As in the classical case, we say π *avoids* σ whenever σ is not a subpermutation of

π, and for any set R of signed permutations we write $B_n(R)$ to denote the set of signed permutations of length n which avoid every pattern in R. For instance, $93\bar{4}72\bar{8}\bar{5}16 \in B_9(\bar{3}2\bar{1}, 1432)$, but $93\bar{4}72\bar{8}\bar{5}16 \notin B_9(31\bar{2})$ since $93\bar{4}72\bar{8}\bar{5}16$ has $92\bar{5}$ as a subsequence. The enumeration of restricted signed permutations was first considered by Simion [10] and has also been studied by Mansour and West [7] and Egge [3], among others. Theorem 2.17 leads to the following enumeration of $B_n(\bar{1}\bar{2}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}, 321)$.

Corollary 2.18 *Fix $n \geq 0$. Then the signed permutation π avoids $\bar{1}\bar{2}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}$, and 321 if and only if π^s avoids 123. In particular, for all $n \geq 0$ we have*

$$|B_n(\bar{1}\bar{2}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}, 321)| = \binom{2n}{n}. \tag{24}$$

Proof. ($\Longleftarrow$) Suppose $\pi \in B_n$; we prove the contrapositive. If $\pi(a)\pi(b)$ is a subsequence of π of type $\bar{1}\bar{2}$ or $\bar{1}2$ then $\pi^s(a)\pi^s(b)\pi^s(2n+1-a)$ is a subsequence of type 123 in π^s. Similarly, if $\pi(a)\pi(b)\pi(c)$ is a subsequence of π of type $\bar{3}2\bar{1}$ or $32\bar{1}$ then $\pi^s(a)\pi^s(b)\pi^s(2n+1-c)$ is a subsequence of type 123 in π^s. Finally, if $\pi(a)\pi(b)\pi(c)$ is a subsequence of π of type $\bar{3}21$ or 321 in π then $\pi^s(a)\pi^s(b)\pi^s(c)$ is a subsequence of type 123 in π^s. Combining these observations we find that if $\pi \in B_n$ contains any of the forbidden patterns $\bar{1}\bar{2}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}$, or 321 then π^s contains 123, as desired.

($\Longrightarrow$) Suppose σ is a subsequence of π^s of type 123; again we prove the contrapositive. Replacing σ with its image under rc if necessary, we may assume the first two entries of σ are among $\pi^s(1), \ldots, \pi^s(n)$. Fix a,b, and c so that $\sigma = \pi^s(a)\pi^s(b)\pi^s(c)$; we consider two cases.

Case One: $\pi^s(c)$ is among $\pi^s(1), \ldots, \pi^s(n)$.

Let α denote the subsequence $\pi(a)\pi(b)\pi(c)$ of π. Note that if an entry of α is unbarred then so are all of the entries to its right, since the sequence $\pi^s(a)\pi^s(b)\pi^s(c)$ could not otherwise have type 123. If all of the entries of α are unbarred then α has type 321, which is forbidden. If only the last two entries of α are unbarred then $\pi(a)\pi(b)$ has type $\bar{1}2$ or α has type $\bar{3}21$, both of which are forbidden. If only the last entry of α is unbarred or if no entry of α is unbarred then $\pi(a)\pi(b)$ has type $\bar{1}\bar{2}$, which is forbidden. In all cases π contains a forbidden subsequence.

Case Two: $\pi^s(c)$ is not among $\pi^s(1), \ldots, \pi^s(n)$.

First note that if $\pi(a)$ and $\pi(b)$ are both barred then $\pi(a)\pi(b)$ has type $\bar{1}\bar{2}$, which is forbidden, and if $\pi(a)$ is unbarred and $\pi(b)$ is barred then $\pi^s(a)\pi^s(b)$ has type 21, which contradicts our assumption that $\pi^s(a)\pi^s(b)\pi^s(c)$ has type 123. Therefore $\pi(b)$ is unbarred. Now observe that if $\pi(a)$ is barred then $\pi(a)\pi(b)$ has type $\bar{2}1$, since $\bar{1}2$ is forbidden, and if $\pi(a)$ is unbarred then $\pi(a)\pi(b)$ has type 21, since $\pi^s(a)\pi^s(b)$ has type 12. Therefore $\pi(a)\pi(b)$ has type $\bar{2}1$ or 21. In either case $\pi(2n+1-c)$ is barred and $\pi(2n+1-c) < \pi(b)$, since $\pi(b)$ is unbarred and $\pi^s(c) > \pi^s(b)$. Therefore the subsequence of π consisting of $\pi(a)$, $\pi(b)$, and $\pi(2n+1-c)$ must be of type $\bar{1}\bar{3}2$, $\bar{3}\bar{1}2$, $\bar{3}2\bar{1}$, $\bar{1}32$, $3\bar{1}2$, or $32\bar{1}$. The third and sixth of these are forbidden, and each of the remaining four contains the subsequence $\bar{1}2$, which is also forbidden.

In both cases above we have shown that if π^s contains a subsequence of type 123 then π contains a forbidden subsequence. Now (24) follows from (20) and Proposition 2.2. □

The fact that $S_{2n}^{rc}(123)$ corresponds to $B_n(\bar{1}\bar{2}, \bar{1}2, \bar{3}2\bar{1}, \bar{3}21, 32\bar{1}, 321)$ under the map s is not an isolated occurrence: for every finite set R of forbidden patterns there is a corresponding finite set T of forbidden signed patterns such that $S_{2n}^{rc}(R)$ corresponds to $B_n(T)$ under s. In order to prove this, we first prove the following lemma.

Lemma 2.19 *Fix $n \geq 0$, suppose $\pi \in B_n$, and let σ denote a subpermutation of π^s. Then there exists a signed permutation α such that the following hold.*

(i) *α is a subpermutation of π.*

(ii) *σ is a subpermutation of α^s.*

(iii) *α^s is a subpermutation of π^s.*

(iv) *The length of α is at most the length of σ.*

Proof. Let ω denote a subsequence of π of type σ, set

$$\Omega = \{j \mid \pi(j) \text{ is in } \omega \text{ or } \pi(2n+1-j) \text{ is in } \omega\},$$

and let β denote the subsequence of π^s consisting of those entries of π^s whose position is in Ω. Now β has even length and its type is invariant under rc by construction, so there exists a signed permutation α such that α^s is the type of β. Observe that (i),(ii), and (iii) hold by our construction of α. Moreover, $|\alpha| = \frac{1}{2}|\beta|$ and $|\beta| \leq 2|\sigma|$ so (iv) also holds. □

Theorem 2.20 *Suppose R is a finite set of permutations. Then there exists a finite set T of signed permutations such that $\pi \in B_n$ avoids T if and only if $\pi^s \in S_{2n}^{rc}$ avoids R.*

Proof. We call a signed permutation π *minimal for R* whenever π^s does not avoid R but if σ is a proper subpermutation of π then σ^s avoids R. Let T denote the set of signed permutations which are minimal for R.

To show that T is finite, observe that if $\pi \in T$ then by Lemma 2.19(i) there is a subpermutation α of π such that α^s does not avoid R. Since π is minimal for R, we must have $\alpha = \pi$. Therefore the length of π is at most the length of the longest element of R by Lemma 2.19(iv). It follows that T is finite.

To complete the proof we show that a signed permutation π has a subpermutation in T if and only if π^s has a subpermutation in R.

($\Longrightarrow$) Suppose $\sigma \in T$ is a subpermutation of π. Then σ^s is a subpermutation of π^s. By the definition of T, the permutation σ^s does not avoid R, so π^s has a subpermutation in R.

($\Longleftarrow$) Suppose $\sigma \in R$ is a subpermutation of π^s and let α be the signed permutation whose existence is guaranteed by Lemma 2.19. By construction α^s has a subpermutation in R. Taking a subpermutation of α, if necessary, we obtain a signed permutation $\beta \in T$. But β is a subpermutation of π, as desired. □

Note that the proof of Theorem 2.20 provides a way to rephrase the proof of Corollary 2.18, since $\{\overline{1}\overline{2}, \overline{1}2, \overline{3}2\overline{1}, \overline{3}21, 32\overline{1}, 321\}$ is the set of signed permutations which are minimal for $R = \{123\}$.

Since we enumerated $S_{2n+1}^{rc}(123, 2143)$ after enumerating $S_{2n+1}^{rc}(123)$, it seems appropriate that we conclude this section by enumerating $S_{2n}^{rc}(123, 2143)$. In fact, we enumerate the permutations in this set according to the position of $2n$.

Theorem 2.21 *For all $n \geq 0$ the following hold.*

(i) *If $\pi \in S_{2n}^{rc}(123, 2143)$ and 1 appears before $2n$ in π then*

$$\pi = 2n-1 \ 2n-2 \ \cdots \ n+2 \ n+1 \ 1 \ 2n \ n \ n-1 \ \cdots \ 32.$$

(ii) *There are exactly F_{2n-2} permutations $\pi \in S_{2n}^{rc}(123, 2143)$ with $\pi(1) = 2n$.*

(iii) *There are exactly $F_{2n-2k+2}$ permutations $\pi \in S_{2n}^{rc}(123, 2143)$ with $\pi(k) = 2n$ for $2 \leq k \leq n$.*

(iv) *We have*

$$|S^{rc}_{2n}(123,2143)| = F_{2n}. \tag{25}$$

Proof. (i) If $\pi \in S^{rc}_{2n}(123,2143)$ and 1 appears before $2n$ in π then by Lemma 2.14 we have $\pi(n) = 1$ and $\pi(n+1) = 2n$ and by Lemma 2.15(ii) the sequence $\pi(1),\ldots,\pi(n-1)$ is decreasing. Moreover, if there exist a,b with $a < n$, $b > n+1$, and $\pi(a) < \pi(b)$ then $\pi(a)\ 1\ 2n\ \pi(b)$ has type 2143, so every entry of π to the left of 1 is greater than every entry to the right of $2n$. Therefore $\pi = 2n-1\ 2n-2\ \cdots\ n+2\ n+1\ 1\ 2n\ n\ n-1\ \cdots\ 32$.

(ii)–(iv) These are immediate for $n = 0$, $n = 1$, and $n = 2$, so we assume $n \geq 3$ and we argue by induction on n. For convenience set $a_n = |S^{rc}_{2n}(123,2143)|$ for all $n \geq 0$ and let Y_k, $1 \leq k \leq n$, denote the set of permutations $\pi \in S^{rc}_{2n}(123,2143)$ with $\pi(k) = 2n$.

To enumerate Y_1 and Y_2, suppose $\pi \in S^{rc}_{2n-2}(123,2143)$ and define π_1 and π_2 by setting

$$\pi_1(j) = \begin{cases} 2n & \text{if } j = 1 \\ \pi(j-1)+1 & \text{if } 2 \leq j \leq 2n-1 \\ 1 & \text{if } j = 2n \end{cases}$$

and

$$\pi_2(j) = \begin{cases} \pi(j)+1 & \text{if } j = 1 \\ 2n & \text{if } j = 2 \\ \pi(j-1)+1 & \text{if } 3 \leq j \leq 2n-2 \\ 1 & \text{if } j = 2n-1 \\ \pi(j-2)+1 & \text{if } j = 2n \end{cases}.$$

Note that by construction $\pi_1, \pi_2 \in S^{rc}_{2n}$. Moreover, since neither of the first two elements of 2143 are 4 and neither of the last two are 1, both π_1 and π_2 avoid 2143. Thus we have bijections between $S^{rc}_{2n-2}(123,2143)$ and each of Y_1 and Y_2, so (ii) and part of (iii) follow from (iv) by induction.

Now fix k with $3 \leq k \leq n$ and suppose $\pi \in Y_k$. By Lemma 2.16(ii) the sequence $\pi(1),\ldots,\pi(k-1)$ is decreasing. Moreover, if there exists j with $1 \leq j \leq k-2$ such that $\pi(j) > \pi(j+1)+1$ then $\pi(j)\pi(j+1)\ 2n\ \pi(j+1)+1$ is a subsequence of type 2143 in π. It follows that $\pi(j) = 2n-j$ for $1 \leq j \leq k-2$. Conversely, it is routine to check that if $\sigma \in S^{rc}_{2n-2k+2}(123,2143)$ and π is given by

$$\pi(j) = \begin{cases} 2n-j & \text{if } 1 \leq j \leq k-2 \\ \sigma(1)+k-1 & \text{if } j = k-1 \\ 2n & \text{if } j = k \\ \sigma(j-k+1)+k-1 & \text{if } k+1 \leq j \leq 2n-k \\ 1 & \text{if } j = 2n-k+1 \\ \sigma(2n-2k+2)+k-1 & \text{if } j = 2n-k+2 \\ 2n-j+2 & \text{if } 2n-k+3 \leq j \leq 2n \end{cases}$$

then $\pi \in S^{rc}_{2n}(123,2143)$. Now (iii) follows from (iv) by induction.

Using (i)–(iii) we now have

$$\begin{aligned} |S^{rc}_{2n}(123,2143)| &= 1 + 2|S^{rc}_{2n-2}(123,2143)| + \sum_{j=3}^{n} |S^{rc}_{2n-2j+2}(123,2143)| \\ &= 2|S^{rc}_{2n-2}(123,2143)| + 1 + \sum_{k=1}^{n-2} |S^{rc}_{2k}(123,2143)| \\ &= 3|S^{rc}_{2n-2}(123,2143)| - |S^{rc}_{2n-4}(123,2143)|. \end{aligned}$$

Now (iv) follows by induction, since it is well-known that $F_{2n} = 3F_{2n-2} - F_{2n-4}$ for all $n \geq 2$. □

3 Involutions Invariant Under the Reverse Complement

Recall from the Introduction that for any $n \geq 0$ and any set R of forbidden patterns, we write $I_n^{rc}(R)$ to denote the set of permutations in S_n which are invariant under $H_2 = \{e, i, irc, rc\}$ and which avoid every pattern in R, and that these are the involutions in $S_n^{rc}(R)$. In this section we find $|I_n^{rc}(R)|$ for various R, beginning with $R = \emptyset$.

Theorem 3.1 *We have* $|I_0^{rc}| = 1$, $|I_2^{rc}| = 2$,

$$|I_{2n+1}^{rc}| = |I_{2n}^{rc}| \qquad (n \geq 0), \tag{26}$$

and

$$|I_{2n}^{rc}| = 2|I_{2n-2}^{rc}| + (2n-2)|I_{2n-4}^{rc}| \qquad (n \geq 2). \tag{27}$$

Proof. It is routine to verify that $\pi \in S_{2n}^{rc}$ is an involution if and only if $\pi^u \in S_{2n+1}^{rc}$ is an involution, and (26) follows.

Now fix $n \geq 2$ and for each j with $1 \leq j \leq 2n$ let A_j denote the set of permutations in I_{2n}^{rc} in which $2n$ occurs in position j. To prove (27) we enumerate $A_1, \ldots, A_{2n}$.

To enumerate A_1, first note that if $\pi \in A_1$ then $\pi(1) = 2n$ and $\pi(2n) = 1$. Moreover, it is clear by considering the diagram of π that the map which sends π to the type of $\pi(2), \ldots, \pi(2n-1)$ is a bijection between A_1 and I_{2n-2}^{rc}. By a similar argument for A_{2n} we conclude that $|A_1| = |A_{2n}| = |I_{2n-2}^{rc}|$.

Now fix j with $2 \leq j \leq 2n-1$ and note that if $\pi \in A_j$ then $\pi(j) = 2n$ and $\pi(2n) = j$, since π is an involution. In addition, $\pi(1) = 2n+1-j$ and $\pi(2n+1-j) = 1$, since $\pi^{rc} = \pi$. By our choice of j we find $1, j, 2n+1-j$, and $2n$ are distinct, so the entries in positions $1, j, 2n+1-j$, and $2n$ of π correspond to the dots in rows and columns $1, j, 2n+1-j$, and $2n$ of the diagram of π. Therefore the map which sends π to the permutation of length $2n-4$ whose diagram is obtained by deleting rows $1, j, 2n+1-j$, and $2n$ and columns $1, j, 2n+1-j$, and $2n$ from the diagram of π is a bijection between A_j and I_{2n-4}^{rc}. It follows that $|A_j| = |I_{2n-4}^{rc}|$.

Since $A_1, \ldots, A_{2n}$ partition I_{2n}^{rc}, when we combine our enumerations above we obtain (27). □

In the table below we give $|I_{2n}^{rc}|$ for $0 \leq n \leq 9$.

n	0	1	2	3	4	5	6	7	8	9
$\|I_{2n}^{rc}\|$	1	2	6	20	76	312	1384	6512	32400	168992

As was the case for S_n^{rc}, and as the results above suggest, there is a close connection between I_n^{rc} and signed permutations. To describe this connection, recall that we may view a signed permutation $\pi \in B_n$ as a permutation of $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}$ by setting $\pi(\overline{j}) = \overline{\pi(j)}$ and $\overline{\overline{j}} = j$ for all j with $1 \leq j \leq n$. We call π a *signed involution* whenever it is an involution when viewed as a permutation of $1, 2, \ldots, n, \overline{1}, \ldots, \overline{n}$.

Theorem 3.2 *Fix $n \geq 0$ and $\pi \in B_n$. Then*

(i) $\pi^s \in I_{2n}^{rc}$ *if and only if π is a signed involution;*

(ii) $\pi^{su} \in I_{2n+1}^{rc}$ *if and only if π is a signed involution.*

Proof. (i) Let $\sigma \in S_{2n}$ denote the permutation of $\{1, 2, \ldots, 2n\}$ induced by π when we identify $\overline{1}, \overline{2}, \ldots, \overline{n}$ with $1, 2, \ldots, n$ respectively and $1, 2, \ldots, n-1, n$ with $2n, 2n-1, \ldots, n+2, n+1$ respectively. By Definition 2.1 we have $\sigma = \pi^s$, and the result follows from the definition of a signed involution.

(ii) This is immediate from (i), since $\pi \in S_{2n}^{rc}$ is an involution if and only if $\pi^u \in S_{2n+1}^{rc}$ is an involution. □

Our goal for the rest of this section is to enumerate $I_n^{rc}(R)$ for various sets of forbidden patterns. As was the case for S_n^{rc}, the symmetry group D_8 allows us to significantly reduce the number of sets R we must consider.

Lemma 3.3 *For any $g \in D_8$ and any set R of patterns, the following hold for all $n \geq 0$.*

(i) *$\pi \in I_n^{rc}(R)$ if and only if $\pi^g \in I_n^{rc}(R^g)$. In particular, $|I_n^{rc}(R)| = |I_n^{rc}(R^g)|$.*

(ii) *$I_n^{rc}(R) = I_n^{rc}(R^{H_2})$.*

Here $R^{H_2} = \{\sigma^h \mid \sigma \in R, h \in H_2\}$ and $R^g = \{\sigma^g \mid \sigma \in R\}$ for all $g \in D_8$.

Proof. (i) Fix $g \in D_8$ and any set R of patterns; by Lemma 2.6(i) it is sufficient to show that if $\pi \in S_n^{rc}$ is an involution then π^g is an involution. Moreover, since r, c, and i together generate D_8, we may assume g is one of these. The result is clear for $g = i$, so suppose $g = r$ and $\pi \in S_n^{rc}$ is an involution. Then we have

$$\begin{aligned} \pi^{ri} &= \pi^{ic} && \text{(since } ri = ic\text{)} \\ &= \pi^c && \text{(since } \pi \text{ is an involution)} \\ &= \pi^r && \text{(since } \pi^{rc} = \pi \text{ and } c^2 = e\text{),} \end{aligned}$$

so π^r is an involution. The proof that π^c is an involution is similar.

(ii) This is similar to the proof of Lemma 2.6(ii). □

By Lemma 3.3(i), the elements of D_8 provide, for all $n \geq 0$, a bijection from $I_n^{rc}(123)$ to $I_n^{rc}(321)$ and bijections among $I_n^{rc}(132)$, $I_n^{rc}(213)$, $I_n^{rc}(231)$, and $I_n^{rc}(312)$. As a result, to enumerate $I_n^{rc}(\sigma)$ for $\sigma \in S_3$ we need only consider $\sigma = 132$ and $\sigma = 123$. We begin with $\sigma = 132$.

Theorem 3.4 *For all $n \geq 0$ we have*

$$S_n^{rc}(132) = I_n^{rc}(132).$$

Proof. Since $I_n^{rc}(132) \subseteq S_n^{rc}(132)$ for all $n \geq 0$, it is sufficient to show that $S_n^{rc}(132) \subseteq I_n^{rc}(132)$ for all $n \geq 0$. To this end, suppose $\pi \in S_n^{rc}(132)$. By Lemmas 2.8 and 2.9 there exist positive integers $l_1, \ldots, l_k$ such that $\pi = \langle l_1, \ldots, l_k \rangle$ and $l_j = l_{k+1-j}$ for all j with $1 \leq j \leq k$. Since $\langle l_1, \ldots, l_k \rangle^i = \langle l_k, \ldots, l_1 \rangle$, we find $\pi^i = \pi$. Therefore $\pi \in I_n^{rc}(132)$, as desired. □

Theorem 3.4 allows us to enumerate $I_n^{rc}(R)$ for any $R \subseteq S_3$ with $132 \in R$.

Corollary 3.5 *We have*

$$|I_{2n}^{rc}(132)| = |I_{2n}^{rc}(132, 213)| = 2^n \qquad (n \geq 0), \tag{28}$$

$$|I_{2n+1}^{rc}(132)| = |I_{2n+1}^{rc}(132, 213)| = 2^n \qquad (n \geq 0), \tag{29}$$

$$|I_n^{rc}(132, 231)| = 2 \qquad (n \geq 2), \tag{30}$$

$$|I_{2n}^{rc}(132, 123)| = |I_{2n+3}^{rc}(132, 123)| = F_{n+1} \qquad (n \geq 0), \tag{31}$$

$$|I_{2n}^{rc}(132, 321)| = 2 \qquad (n \geq 1), \tag{32}$$

$$|I_{2n+1}^{rc}(132, 321)| = 1 \qquad (n \geq 0), \tag{33}$$

and

$$|I_n^{rc}(132, 123, 231)| = |I_n^{rc}(132, 321, 231)| = 1 \qquad (n \geq 3). \tag{34}$$

Proof. This is immediate from Theorems 3.4 and 2.10. □

Suppose R_1 and R_2 are sets of permutations. We say R_1 and R_2 are H_2-*Wilf equivalent* whenever there exists N such that $|I_n^{rc}(R_1)| = |I_n^{rc}(R_2)|$ for all $n \geq N$. By Lemma 3.3, most subsets of S_3 are H_2-Wilf equivalent to a set of forbidden patterns in Corollary 3.5. The exceptions are $\{123\}$ and those sets which contain $\{123, 321\}$. If a set R is of the latter type then then by the Erdös-Szekeres Theorem we have $|I_n(R)| = 0$ for all $n \geq 5$. With these observations in mind, we turn our attention to $I_n^{rc}(123)$, beginning with the case in which n is odd.

Theorem 3.6 *For all $n \geq 0$ we have*

$$|I_{2n+1}^{rc}(123)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{35}$$

Proof. By Theorem 3.2(ii) and Lemma 2.11 the map su is a bijection between $I_{2n+1}^{rc}(123)$ and the set of involutions in S_n which avoid 123. Now the result follows from [11, Prop. 3]. □

In order to enumerate $I_n^{rc}(123)$ when n is even we need a technical result concerning the structure of the permutations in $I_{2n}^{rc}(123)$.

Lemma 3.7 *Suppose $\pi \in I_{2n}^{rc}$ has $\pi(k) = 2n$, where $1 < k < n+1$. Let σ denote the permutation in I_{2n-4}^{rc} whose diagram is obtained by removing rows $1, k, 2n+1-k$, and $2n$ and columns $1, k, 2n+1-k$, and $2n$ from the diagram of π, and suppose σ avoids 123. Then the following are equivalent.*

(i) *123 is a subpermutation of π.*

(ii) *There exists j with $1 \leq j \leq k-2$ such that $\sigma(j) \geq 2n-k-2$.*

(iii) *If m has $\sigma(m) = 2n-4$ then $m \leq k-2$.*

Proof. (i) $\Longrightarrow$ (ii) Fix a subsequence ω of π of type 123. Since σ avoids 123 the subsequence ω must involve at least one of $\pi(1), \pi(k), \pi(2n+1-k)$, and $\pi(2n)$. Replace ω with its image under an appropriate element of H_2 so that ω involves $\pi(k)$. Since $\pi(k) = 2n$, we see that $\pi(k)$ must play the role of the 3 in ω. If $\pi(1)$ plays the role of 1 then the position of the second element of ω is the desired j. Otherwise σ has an increase among $\sigma(1), \ldots, \sigma(k-2)$; let $\sigma(a)\sigma(b)$ denote one such increase. If $\sigma(b) = 2n-4$ then (ii) holds with $j = b$. If $\sigma(b) \neq 2n-4$ then $2n-4$ appears to the left of $\sigma(b)$, since σ avoids 123, and again (ii) holds.

(ii) $\Longrightarrow$ (iii) If $m > j$ then $\sigma(1)\sigma(j)\sigma(m)$ has type 123, since $\sigma(1) = 2n-3-k$. But σ avoids 123 so we must have $m \leq j \leq k-2$.

(iii) $\Longrightarrow$ (i) This is immediate, since $\pi(1)\pi(m+1)\pi(k)$ has type 123. □

Using Lemma 3.7 we obtain the following refined enumeration of $I_{2n}^{rc}(123)$.

Theorem 3.8 *Fix $n \geq 1$ and k with $1 \leq k \leq 2n$. Then the following hold.*

(i) *If $k > n+1$ then no permutation $\pi \in I_{2n}^{rc}(123)$ has $\pi(k) = 2n$.*

(ii) *If $\pi \in I_{2n}^{rc}(123)$ and $\pi(n+1) = 2n$ then $\pi = n\ n-1\ \ldots 2\ 1\ 2n\ 2n-1\ \ldots n+1$.*

(iii) *There are exactly 2^{n-k} permutations $\pi \in I_{2n}^{rc}(123)$ with $\pi(k) = 2n$ for $1 \leq k \leq n$.*

(iv) *We have*

$$|I_{2n}^{rc}(123)| = 2^n. \tag{36}$$

Proof. (i) If $\pi \in I_{2n}^{rc}$ has $\pi(k) = 2n$ and $k > n+1$ then $2n+1-k < n$ and $\pi(2n+1-k) = 1$ so $\pi(2n+1-k)\pi(n+1)\pi(k)$ has type 123.

(ii) Suppose $\pi \in I_{2n}^{rc}(123)$ has $\pi(n+1) = 2n$ and observe that $\pi(2n) = n+1$, $\pi(n) = 1$, and $\pi(1) = n$. Now if one of $n+2, \ldots, 2n-1$ appears among $\pi(2), \ldots, \pi(n-1)$ then $\pi(1), \pi(n+1)$, and this entry form a subsequence of type 123, so the entries $\pi(n+2), \ldots, \pi(2n-1)$ must be $n+2, \ldots, 2n-1$. Moreover, if $\pi(n+2), \ldots, \pi(2n-1)$ contains an increase then $\pi(n)$ together with this increase has type 123, so $\pi(j) = 3n+1-j$ for all j with $n+2 \leq j \leq 2n-1$. Since $\pi \in I_{2n}^{rc}$ we must also have $\pi(j) = n+1-j$ for all j with $2 \leq j \leq n-1$. Therefore $\pi = n\, n{-}1\, \ldots 2\, 1\, 2n\, 2n{-}1\, \ldots n{+}1$, and it is routine to check that π does in fact avoid 123.

(iii),(iv) Both (iii) and (iv) are immediate for $n = 1$ and $n = 2$ so we assume $n \geq 3$ and we argue by induction on n.

First suppose $k > 1$. The map defined in Lemma 3.7 is a bijection between the set of $\pi \in I_{2n}^{rc}(123)$ with $\pi(k) = 2n$ and the set of $\sigma \in I_{2n-4}^{rc}(123)$ in which $2n-4$ appears in one of the positions $k-1, \ldots, n-1$. By (ii) there is one such σ in which $2n-4$ appears in position $n-1$. By induction there are 2^j such σ in which $2n-4$ appears in position $n-j-2$ for $0 \leq j \leq n-1-k$. Therefore there are $1 + \sum_{j=0}^{n-1-k} 2^j = 2^{n-k}$ elements $\pi \in I_{2n}^{rc}(123)$ with $\pi(k) = 2n$, and (iii) follows.

Now suppose $k = 1$. In this case $\pi(1) = 2n$ so $\pi(2n) = 1$. For each such π, let π' denote the permutation whose diagram is obtained from the diagram of π by removing rows 1 and $2n$ and columns 1 and $2n$. It is routine to show that the map $\pi \mapsto \pi'$ is a bijection between $I_{2n-2}^{rc}(123)$ and the set of $\pi \in I_{2n}^{rc}(123)$ with $\pi(1) = 2n$. By induction this set has exactly 2^{n-1} elements and (iii) follows.

Using (i),(ii), and (iii) we find

$$\begin{aligned} |I_{2n}^{rc}(123)| &= 1 + 2^{n-1} + \sum_{k=2}^{n} 2^{n-k} \\ &= 2^n, \end{aligned}$$

and (iv) follows. □

Theorem 3.8 completes our analysis of the H_2-Wilf-equivalence classes of subsets of S_3. In the table below we summarize the H_2-Wilf-equivalence classes for which $|I_n^{rc}(R)| \neq 0$.

$\lvert R\rvert$	R	$\lvert I_n^{rc}(R)\rvert$
0	$\emptyset$	Equations (26) and (27)
1	$\{123\}$	Equations (35) and (36)
	$\{132\}$	Equations (28) and (29)
2	$\{132, 213\}$	Equations (28) and (29)
	$\{132, 231\}$	Equation (30)
	$\{132, 123\}$	Equation (31)
	$\{132, 321\}$	Equations (32) and (33)
3	$\{132, 123, 231\}$	Equation (34)

H_2-Wilf-equivalence classes for $R \subseteq S_3$

In Corollary 2.18 we saw that the enumeration of $S_{2n}^{rc}(123)$ leads to an enumeration of a certain set of pattern-avoiding signed permutations. In a similar way, Theorem 3.8 leads to the following enumeration of a certain set of pattern-avoiding signed involutions.

Corollary 3.9 *Fix $n \geq 1$ and k with $1 \leq k \leq n$. Then the following hold.*

(i) *If $\pi \in B_n(\overline{1}\overline{2}, \overline{1}2, \overline{3}2\overline{1}, \overline{3}21, 32\overline{1}, 321)$ is a signed involution in which 1 is barred then $\pi = \overline{n}\, \overline{n-1}\, \overline{n-2} \cdots \overline{2}\, \overline{1}$.*

(ii) *There are exactly* 2^{n-k} *signed involutions* $\pi \in B_n(\overline{1}\overline{2}, \overline{1}2, \overline{3}2\overline{1}, \overline{3}21, 32\overline{1}, 321)$ *in which* 1 *is unbarred and* $\pi(k) = 1$.

(iii) *There are exactly* 2^n *signed involutions in* $B_n(\overline{1}\overline{2}, \overline{1}2, \overline{3}2\overline{1}, \overline{3}21, 32\overline{1}, 321)$.

Proof. This is immediate from Corollary 2.18 and Theorems 3.2 and 3.8. □

As was the case for S_{2n}^{rc}, the correspondence exemplified in Corollary 3.9 holds more generally.

Theorem 3.10 *Suppose* R *is a finite set of permutations. Then there exists a finite set* T *of signed permutations such that a signed involution* $\pi \in B_n$ *avoids* T *if and only if* $\pi^s \in I_{2n}^{rc}$ *avoids* R.

Proof. This is immediate from Theorems 2.20 and 3.2. □

4 Rotationally Symmetric Permutations

Recall from the Introduction that for any $n \geq 0$ and any set R of forbidden patterns we write $S_n^{90}(R)$ to denote the set of permutations in S_n which are invariant under $H_3 = \{e, rc, ri, ci\}$ and which avoid every pattern in R. We observe that the four elements of H_3 act as rotations by multiples of $\frac{\pi}{2}$ on diagrams of permutations. In particular, if $\pi \in S_n^{90}$ has $\pi(j) = k$ then $\pi(n+1-k) = j$, $\pi(n+1-j) = n+1-k$, and $\pi(k) = n+1-j$. In this section we determine $|S_n^{90}(R)|$ for various R, beginning with $R = \emptyset$.

Theorem 4.1 *For all* $n \geq 0$ *we have*

$$|S_{4n}^{90}| = 2^n(2n-1)!!, \tag{37}$$

$$|S_{4n+1}^{90}| = |S_{4n}^{90}|, \tag{38}$$

and

$$|S_{4n+2}^{90}| = |S_{4n+3}^{90}| = 0. \tag{39}$$

Proof. It is routine to show that $\pi \in S_{2n}^{90}$ if and only if $\pi^u \in S_{2n+1}^{90}$, from which (38) and part of (39) follow.

To complete the proof of (39) suppose $\pi \in S_{4n+2}^{90}$. Since $|\pi|$ is not divisible by 4, there exists j with $1 \leq j \leq 4n+2$ such that $j, k = \pi(j), 4n+3-j, 4n+3-k$ are not distinct. Without loss of generality, suppose j is equal to k, $4n+3-j$, or $4n+3-k$. If $j = 4n+3-j$ then $2j = 4n+3$, which is a contradiction. If $j = k$ then $j = \pi(j) = 4n+3-j$, which is another contradiction. If $j = 4n+3-k$ then $k = \pi(j) = \pi(4n+3-k) = j$, which we have seen is impossible. Therefore both expressions in (39) must be zero.

To prove (37), first observe that the result is immediate for $n = 0$ and $n = 1$, so we argue by induction on n. For each j with $1 \leq j \leq 4n$, let A_j denote the set of permutations in S_{4n}^{90} with $\pi(j) = 4n$. Note that if $\pi(1) = 1$ then $\pi(4n) = 1$, which is a contradiction, so A_1 is empty. By a similar argument A_{4n} is also empty. It is routine to verify that the map which sends $\pi \in S_{4n}^{90}$ to the permutation whose diagram is obtained by removing rows $1, j, 4n+1-j$, and $4n$ and columns $1, j, 4n+1-j$, and $4n$ from the diagram of π is a bijection between A_j and S_{4n-4}^{90} for all j with $2 \leq j \leq 4n-1$. Therefore $|S_{4n}^{90}| = (4n-2)|S_{4n-4}^{90}|$ and (37) follows by induction. □

Before we turn our attention to $S_n^{90}(R)$ for nonempty R, we digress briefly to describe how the permutations in S_n^{90} are connected with signed permutations. To set our notation for this result, suppose $\pi \in B_n$. We write $\overline{\pi}$ to denote the signed permutation in B_n obtained by replacing each barred entry of π with its unbarred counterpart and vice versa, and we write π^{-1} to denote the group-theoretic inverse of π. For example, if $\pi = \overline{3}41\overline{5}2$ then $\overline{\pi} = 3\overline{4}\overline{1}5\overline{2}$, $\pi^{-1} = 35\overline{1}2\overline{4}$, and $\overline{\pi}^{-1} = \overline{3}\overline{5}1\overline{2}4$.

Theorem 4.2 *The following hold for all $n \geq 0$.*

(i) *For all $\pi \in B_n$ we have $\pi^{scis^{-1}} = \overline{\pi}^{-1}$.*

(ii) *For all $\pi \in B_n$ we have $\pi^s \in S_{2n}^{90}$ if and only if $\overline{\pi}^{-1} = \pi$.*

Proof. (i) Fix j with $1 \leq j \leq n$ and suppose $\pi \in B_n$ has $\pi(k) = \overline{j}$. Then $\pi^s(2n+1-k) = 2n+1-j$ and $\pi^{sci}(j) = 2n+1-k$, so $\pi^{scis^{-1}}(j) = k = \overline{\pi}^{-1}(j)$. Similarly, if $\pi(k) = j$ then $\pi^s(k) = 2n+1-j$ and $\pi^{sci}(j) = k$, so $\pi^{scis^{-1}}(j) = \overline{k} = \overline{\pi}^{-1}(j)$.

(ii) Since ci generates H_3 it is sufficient to show that π^s is invariant under ci if and only if $\overline{\pi}^{-1} = \pi$.

($\Longrightarrow$) If π^s is invariant under ci then we have

$$\begin{aligned} \overline{\pi}^{-1} &= \pi^{scis^{-1}} && \text{(by (i))} \\ &= \pi^{ss^{-1}} && \text{(since } \pi^s \text{ is invariant under } ci) \\ &= \pi, \end{aligned}$$

as desired.

($\Longleftarrow$) If $\overline{\pi}^{-1} = \pi$ then we have

$$\begin{aligned} \pi^{sci} &= \pi^{scis^{-1}s} \\ &= (\overline{\pi}^{-1})^s && \text{(by (i))} \\ &= \pi^s && \text{(since } \overline{\pi}^{-1} = \pi), \end{aligned}$$

as desired. □

It is routine to check directly that no signed permutations of odd length satisfy $\overline{\pi}^{-1} = \pi$; Theorem 4.2 allows us to count the signed permutations of even length with this symmetry.

Corollary 4.3 *There are exactly $2^n(2n-1)!!$ signed permutations $\pi \in B_{2n}$ with $\overline{\pi}^{-1} = \pi$.*

Proof. This is immediate from Theorem 4.2(ii) and (37). □

Our goal for the rest of this section is to enumerate $S_n^{90}(R)$ for various sets of forbidden patterns. As was the case for S_n^{rc} and I_n^{rc}, the symmetry group D_8 allows us to significantly reduce the number of sets R we must consider.

Lemma 4.4 *For any $g \in D_8$ and any set R of patterns, the following hold for all $n \geq 0$.*

(i) *$\pi \in S_n^{90}(R)$ if and only if $\pi^g \in S_n^{90}(R^g)$. In particular, $|S_n^{90}(R)| = |S_n^{90}(R^g)|$.*

(ii) *$S_n^{90}(R) = S_n^{90}(R^{H_3})$.*

Here $R^{H_3} = \{\sigma^h \mid \sigma \in R, h \in H_3\}$ and $R^g = \{\sigma^g \mid \sigma \in R\}$ for all $g \in D_8$.

Proof. (i) First note that the result is immediate for $g \in H_3$. Now since H_3 together with i generate D_8, it is sufficient to prove the result for $g = i$. Moreover, H_3 is generated by ri, so it is sufficient to prove that if $\pi \in S_n^{90}$ then $\pi^{iri} = \pi^i$. To this end we note that $\pi^{iri} = \pi^{cii}$ since $ir = ci$ and $\pi^{cii} = \pi^i$ since $\pi^{ci} = \pi$, and the result follows.

(ii) This is similar to the proof of Lemma 2.6(ii). □

By Lemma 4.4(i), the elements of D_8 provide, for all $n \geq 0$, a bijection from $S_n^{90}(123)$ to $S_n^{90}(321)$ and bijections among $S_n^{90}(132)$, $S_n^{90}(213)$, $S_n^{90}(231)$, and $S_n^{90}(312)$. Consequently, to enumerate $S_n^{90}(\sigma)$ for $\sigma \in S_3$ we need only consider $\sigma = 132$ and $\sigma = 123$. As we see next, this is actually sufficient to enumerate $S_n^{90}(R)$ for all R which contain an element of S_3.

Theorem 4.5 *We have*

$$|S_n^{90}(132)| = 0 \qquad (n \geq 2), \tag{40}$$

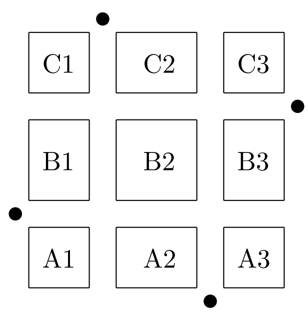
$$|S_{2n+1}^{90}(123)| = 0 \qquad (n \geq 1), \tag{41}$$

and

$$|S_{4n}^{90}(123)| = 0 \qquad (n \geq 2). \tag{42}$$

Proof. To prove (40), first note that the result follows from Theorem 4.1 for $n = 2$ and $n = 3$, so suppose $n \geq 4$. Now observe that if $\pi \in S_n^{90}$ and $j = \pi(1)$ then $1, j, n$, and $n + 1 - j$ are distinct. Let σ denote the permutation in S_4^{90} obtained by removing all rows and columns except those numbered $1, j, n, n + 1 - j$ from the diagram of π. Then $\sigma = 2413$ or $\sigma = 3142$, each of which has 132 as a subpermutation. Therefore π has 132 as a subpermutation and (40) follows.

To prove (42), suppose $\pi \in S_{4n}^{90}$ and $\pi(1) \leq 2n$. Then the diagram of π has the following form.



If π avoids 123 then regions C1, B1, B2, B3, and A3 must be empty. Since the diagram of π is invariant under rotations by multiples of $\frac{\pi}{2}$, all of the remaining regions are empty as well, so $\pi \in S_4^{90}$. The case in which $\pi(1) \geq 2n + 1$ is similar.

The proof of (41) is similar to the proof of (42). □

Corollary 4.6 *If R contains a permutation of length 3 or less then $|S_n^{90}(R)| = 0$ for all $n \geq 5$.*

Proof. This is immediate from Theorem 4.1, Lemma 4.4, and Theorem 4.5. □

Having enumerated $S_n^{90}(R)$ for all sets R of forbidden patterns which contain a permutation of length 3 or less, we now turn our attention to $S_n^{90}(\sigma)$ for $\sigma \in S_4$. We say sets R_1 and R_2 of permutations are *H_3-Wilf equivalent* whenever there exists N such that $|S_n^{90}(R_1)| = |S_n^{90}(R_2)|$ for all $n \geq N$. By Lemma 4.4, every permutation in S_4 is H_3-Wilf equivalent to 1234, 1243, 1324, 1342, 1432, 2143, or 2413. Moreover, by Lemma 4.4(ii) we have $S_n^{90}(1234) = S_n^{90}(1234, 4321)$, which is empty for all $n \geq 10$ by the Erdös-Szekeres Theorem. As we show next, $S_n^{90}(1243)$ and $S_n^{90}(1432)$ are also empty for large values of n.

Theorem 4.7 *For all $n \geq 6$ we have*

$$|S_n^{90}(1243)| = |S_n^{90}(1432)| = 0. \tag{43}$$

Proof. First note that (43) is immediate from (39) for $n = 6$ and $n = 7$. Moreover, inspection of S_8^{90} reveals that (43) holds for $n = 8$. Now observe that if $\pi \in S_{2k+1}^{90}(R)$ and σ is the permutation whose diagram is obtained by removing row $k+1$ and column $k+1$ from the diagram of π then $\sigma \in S_{2k}^{90}(R)$. Similarly, if $\pi \in S_{4k}^{90}(R)$ has $\pi(j) = 4k$ and σ is the permutation whose diagram is obtained by removing rows $1, j, 4k+1-j$, and $4k$ and columns $1, j, 4k+1-j$, and $4k$ from the diagram of π then $\sigma \in S_{4k-4}^{90}(R)$. Therefore (43) holds by induction on n. □

Although $S_n^{90}(1342)$ is not quite empty, as we show next the permutations it contains can have one of only two forms.

Theorem 4.8 *The following hold for all $n \geq 1$.*

(i) *If $\pi \in S_{4n}^{90}(1342)$ then π consists of four consecutive monotone sequences of length n, and in particular*

$$\pi = 2n+1, \ldots, 3n, n, \ldots, 1, 4n, \ldots, 3n+1, n+1, \ldots, 2n$$

or

$$\pi = n+1, \ldots, 2n, 4n, \ldots, 3n+1, n, \ldots, 1, 2n+1, \ldots, 3n.$$

(ii) *If $\pi \in S_{4n+1}^{90}(1342)$ then*

$$\pi = n+1, \ldots, 2n, 4n+1, \ldots, 3n+2, 2n+1, n, \ldots, 1, 2n+2, \ldots, 3n+1.$$

Proof. (i) The result is immediate for $n = 1$ so we assume $n \geq 2$ and we argue by induction on n. Fix $\pi \in S_{4n}^{90}(1342)$ with $\pi(j) = 4n$, let σ denote the permutation whose diagram is obtained by removing rows $1, j, 4n+1-j$, and $4n$ and columns $1, j, 4n+1-j$, and $4n$ from the diagram of π, and observe that $\sigma \in S_{4n-4}^{90}(1342)$. By induction we have

$$\sigma = 2n-1, \ldots, 3n-3, n-1, \ldots, 1, 4n-4, \ldots, 3n-2, n, \ldots, 2n-2$$

or

$$\sigma = n, \ldots, 2n-2, 4n-4, \ldots, 3n-2, n-1, \ldots, 1, 2n-1, \ldots, 3n-3.$$

One easily checks that in the first case π avoids 1342 if and only if $\pi(1) = 2n+1$ and in the second case π avoids 1342 if and only if $\pi(1) = n+1$. This determines π and (i) follows.

(ii) If $\pi \in S_{4n+1}^{90}(1342)$ then $\pi(2n+1) = 2n+1$ and the permutation whose diagram is obtained by removing row $2n+1$ and column $2n+1$ from the diagram of π is in $S_{4n}^{90}(1342)$. Now (ii) follows from (i). □

We now consider $S_n^{90}(1324)$. In order to enumerate this set, we first study its recursive structure.

Lemma 4.9 *For all $n \geq 1$ if $\pi \in S_{4n}^{90}(1324)$ then exactly one of the following holds.*

(i) *$\pi(2n) = 4n$ and $\pi(2n+2) = 4n-1$.*

(ii) *$\pi(2n+1) = 4n$ and $\pi(2n-1) = 4n-1$.*

(iii) *$4n$ is adjacent to $4n-1$ in π.*

Proof. Conditions (i)-(iii) are clearly mutually exclusive, so at most one of them holds.

To show that at least one of (i)-(iii) holds suppose $\pi \in S_{4n}^{90}$ has $\pi(a) = 4n$, $\pi(b) = 4n-1$, and π satisfies none of (i)-(iii). Since $\pi(4n+1-a) = 1$ the only way 1 and $4n$ can be adjacent is if $a = 2n$ or $a = 2n+1$. Therefore there exists c between a and b with $j = \pi(c) > 1$. Now remove all rows and columns from the diagram of π except those numbered $1, a, 4n+1-a, 4n$, those numbered $2, b, 4n+1-b, 4n-1$, and those numbered $j, c, 4n+1-j, 4n+1-c$ to obtain the diagram of a permutation π'. Note that if one of $j, c, 4n+1-j, 4n+1-c$ is 2 then $\pi' \in S_8^{90}$; otherwise $\pi' \in S_{12}^{90}$. Moreover, observe that π' does not satisfy any of (i)-(iii). It is routine to check that every such π' has 1324 as a subpermutation, so 1324 is a subpermutation of π. □

We have seen that for any permutation $\pi \in S_{4n-4}^{90}$ and any j with $2 \leq j \leq 4n-1$ there exists a unique permutation π' with $\pi'(j) = 4n$ such that the diagram obtained by removing rows and columns $1, j, 4n+1-j$, and $4n$ from the diagram of π' is the diagram of π. In this context we refer to π' as the *jth child of* π. In our next lemma we show that if $\pi \in S_{4n-4}^{90}$ avoids 1324 then all of the children of π which satisfy one of Lemma 4.9(i)-(iii) also avoid 1324.

Lemma 4.10 *For all $n \geq 1$ and all $\pi \in S_{4n-4}^{90}(1324)$ the following hold.*

(i) *If $\pi(2n-2) = 4n-4$ then the $(2n-1)$th, $(2n)$th, and $(2n+1)$th children of π all avoid 1324.*

(ii) *If $\pi(2n-1) = 4n-4$ then the $(2n)$th, $(2n+1)$th, and $(2n+2)$th children of π all avoid 1324.*

(iii) *If $\pi(j) = 4n-4$, where $2 < j < 2n-2$ or $2n-1 < j < 4n-5$, then the two children of π in which $4n$ is adjacent to $4n-1$ both avoid 1324.*

Proof. (i) Suppose by way of contradiction that π' is the $(2n+1)$th child of π and π' contains a subsequence σ of type 1324. Since π avoids 1324, the subsequence σ must involve at least one of $1, 2n, 2n+1$, and $4n$. Replace σ with its image under an element of H_3 if necessary, so that σ involves $\pi'(2n+1) = 4n$. In this case $4n$ plays the role of the 4 in σ and neither $\pi'(2n) = 1$ nor $\pi'(2n-1) = 4n-1$ can be involved in σ. Therefore the subsequence obtained by replacing $4n$ with $\pi'(2n-1) = 4n-1$ in σ is a subsequence of type 1324 or 4231 in π. Repeating this process if necessary, we eventually obtain a subsequence of type 1324 or 4231 in π. Applying an appropriate element of H_3, we obtain a subsequence of type 1324 in π, which is a contradiction. Therefore π' avoids 1324. The remaining cases are similar.

(ii),(iii) These are similar to the proof of (i). □

Lemmas 4.9 and 4.10 lead us to the following refined enumeration of $S_{4n}^{90}(1324)$.

Theorem 4.11 *For all $n \geq 1$ the following hold.*

(i) *There are exactly $\binom{n-1}{3n-m-k}$ permutations $\pi \in S_{4n}^{90}(1324)$ with $\pi(k) = 4n$ and $|\{j \mid 1 \leq j \leq n$ and $\pi(j) < 2n+1\}| = m$ for $2 \leq k \leq 4n-1$ and $0 \leq m \leq n$.*

(ii) *We have*

$$|S_{4n}^{90}(1324)| = (n+1)2^{n-1}. \tag{44}$$

Proof. (i) The result is routine to check for $n = 1$ and $n = 2$ so we assume $n \geq 3$ and we argue by induction on n. For convenience, let $A(n,k,m)$ denote the set of permutations in $S_{4n}^{90}(1324)$ with $\pi(k) = 4n$ and $|\{j \mid 1 \leq j \leq n$ and $\pi(j) < 2n+1\}| = m$. Also, set $a(n,k,m) = |A(n,k,m)|$.

Suppose $k = 2n$. By Lemma 4.9(i),(iii), if $\pi \in A(n,2n,m)$ then $\pi(2n-1) = 4n-1$ or $\pi(2n+2) = 4n-1$. By Lemma 4.10 the map which acts by removing rows and columns $1, 2n, 2n+1, 4n$ from the diagram of π is a bijection between the set of those $\pi \in A(n,2n,m)$ with $\pi(2n-1) = 4n-1$ and $A(n-1, 2n-2, m-1)$. Moreover, this map is also a bijection between the set of those $\pi \in A(n,2n,m)$ with $\pi(2n+2) = 4n-1$ and $A(n-1, 2n-1, m-1)$. Therefore by induction we have

$$\begin{aligned} A(n,2n,m) &= A(n-1,2n-2,m-1) + A(n-1,2n-1,m-1) \\ &= \binom{n-2}{n-m} + \binom{n-2}{n-m-1} \end{aligned}$$

$$\begin{aligned} &= \binom{n-1}{n-m} \\ &= \binom{n-1}{3n-k-m}, \end{aligned}$$

as desired.

The proofs for $k = 2n+1$, $k > 2n+1$, and $k < 2n$ are similar to the proof for $k = 2n$.

(ii) By (i) and Lemma 4.9 we have

$$\begin{aligned} |S^{90}_{4n}(1324)| &= \sum_{m=0}^{n} \sum_{k=2}^{4n-1} \binom{n-1}{3n-k-m} \\ &= \sum_{m=0}^{n} 2^{n-1} \\ &= (n+1)2^{n-1}, \end{aligned}$$

as desired. □

To enumerate $S^{90}_{4n+1}(1324)$ we first determine for which $\pi \in S^{90}_{4n}(1324)$ the permutation $\pi^u \in S^{90}_{4n+1}$ avoids 1324.

Lemma 4.12 *Fix $n \geq 1$ and $\pi \in S^{90}_{4n}(1324)$ and recall that $\pi^u \in S^{90}_{4n+1}$. Then the following are equivalent.*

(i) *π^u avoids 1324.*

(ii) *$\pi(j) > 2n$ for all j with $1 \leq j \leq n$ or $\pi(j) < 2n+1$ for all j with $1 \leq j \leq n$.*

Proof. As in the proof of Lemma 4.9, if there is a counterexample then there is a counterexample of length 13 or less. Inspection of S^{90}_{13}, S^{90}_{9}, and S^{90}_{5} shows that no such counterexample exists. □

Theorem 4.13 *For all $n \geq 1$ the following hold.*

(i) *There are exactly $\binom{n-1}{3n-k+1} + \binom{n-1}{2n-k}$ permutations $\pi \in S^{90}_{4n+1}(1324)$ with $\pi(k) = 4n+1$ for $2 \leq k \leq 4n$.*

(ii) *We have*

$$|S^{90}_{4n+1}(1324)| = 2^n. \tag{45}$$

Here we use the convention that $\binom{a}{b} = 0$ whenever $a < 0$, $b < 0$, or $a < b$.

Proof. (i) This is immediate from Theorem 4.11 and Lemma 4.12.

(ii) This is similar to the proof of Theorem 4.11(i), using (i). □

We now turn our attention to $S^{90}_{4n}(2143)$. In order to enumerate this set, we determine where the entries $4n$ and $4n-1$ can occur in a permutation $\pi \in S^{90}_{4n}(2143)$. To begin, we show that it is sufficient to consider those permutations in which $4n$ is in the left half.

Lemma 4.14 *Fix $n \geq 1$ and k with $1 \leq k \leq 4n$. Then the number of permutations $\pi \in S^{90}_{4n}(2143)$ with $\pi(k) = 4n$ is equal to the number of permutations $\pi \in S^{90}_{4n}(2143)$ with $\pi(4n+1-k) = 4n$.*

Proof. Note that $2143^{ir} = 3412 = 2143^r$, so $S^{90}_{4n}(2143)$ is closed under r by Lemma 4.4(i). Therefore r is a bijection between the two sets in question. □

The entry $4n$ cannot occur first in a permutation $\pi \in S^{90}_{4n}$, and as we show next, those $\pi \in S^{90}_{4n}(2143)$ in which $4n$ occurs second are in bijection with $S^{90}_{4n-4}(2143)$.

Lemma 4.15 *Fix $n \geq 1$, suppose $\pi \in S_{4n}^{90}$ has $\pi(2) = 4n$. Let $\sigma \in S_{4n-4}^{90}$ denote the permutation whose diagram is obtained by removing rows and columns $1, 2, 4n-1$, and $4n$ from the diagram of π. Then π avoids 2143 if and only if σ avoids 2143.*

Proof. If π avoids 2143 then σ avoids 2143, so suppose π contains a subsequence α of type 2143. Observe that $\pi(2) = 4n$ cannot be involved in α since the 4 appears third in 2143 and $\pi(4n-1) = 1$ cannot be involved in α since the 1 appears second in 2143. Therefore $\pi(4n) = 4n - 1$ cannot be involved in α since it would have to play the role of the 4 and $\pi(1) = 2$ cannot be involved in α since it would have to play the role of the 1. Hence none of $1, 2, 4n-1$, and $4n$ are involved in α, so α is a subsequence of σ. Therefore σ does not avoid 2143. □

Finally, we characterize those $\pi \in S_{4n}^{90}(2143)$ in which $4n$ occurs in one of positions 3 through $2n$ according to where $4n-1$ can be in relation to $4n$.

Lemma 4.16 *Fix $\pi \in S_{4n}^{90}$ with $\pi(k) = 4n$ and $\pi(j) = 4n-1$. Let π' denote the permutation in S_{4n-4}^{90} whose diagram is obtained by deleting rows and columns $1, k, 4n+1-k$, and $4n$ from the diagram of π. Moreover, suppose $3 \leq k \leq 2n$ and π' avoids 2143. Then π avoids 2143 if and only if $k < j \leq 2n$.*

Proof. ($\Longrightarrow$) First note that if $j \geq 2n+1$ then $\pi(1)\ 2\ (4n-1)\ \pi(4n)$ has type 2143, which is a contradiction. Similarly, if $j < k \leq 2n$ then $\pi(1)\pi(2)(4n-1)\pi(4n)$ has type 2143, which is also a contradiction.

($\Longleftarrow$) It is routine to verify the result for $n = 1$ and $n = 2$ so we assume $n \geq 3$ and we argue by induction on n.

Suppose the entries in positions $a < b < c < d$ of π form a subsequence α of type 2143. By hypothesis $4n$ is to the left of $4n-1$ and by induction $4n-1$ is to the left of $4n-2$, so if $c = k$ then $\pi(a)\pi(b)\ 4n-1\ 4n-2$ is also a subsequence of π of type 2143. Since this is the only way $4n$ can be involved in α, we may assume $4n$ is not involved in α. By a similar argument we may also assume 1 is not involved in α. Now if $\pi(1)$ is involved in α then $\pi(b)$ is among $\pi(2n+1), \ldots, 1$ since π' avoids 2143. Therefore $\pi(c)$ is to the right of 1 in π and hence $\pi(c) > 2n$. It follows that $\pi(2)\pi(b)\pi(c)\pi(d)$ also has type 2143 so we may assume $\pi(1)$ is not involved in α. By a similar argument we may assume $\pi(4n)$ is also not involved in α. Therefore if π has a subsequence of type 2143 then so does π', and the result follows. □

Lemmas 4.14–4.16 lead us to the following refined enumeration of $S_{4n}(2143)$.

Theorem 4.17 *The following hold for all $n \geq 1$.*

(i) *There are exactly $\binom{2n-k}{n-k+1}$ permutations $\pi \in S_{4n}^{90}(2143)$ with $\pi(k) = 4n$ for $2 \leq k \leq 2n$.*

(ii) *We have*

$$|S_{4n}^{90}(2143)| = \binom{2n}{n}. \tag{46}$$

Proof. Both (i) and (ii) are immediate for $n = 1$ and $n = 2$, so we assume $n \geq 3$ and we argue by induction on n.

In view of Lemma 4.15, when $k = 2$ line (i) follows from (ii) by induction. Now let $a_{n,k}$ denote the number of permutations $\pi \in S_{4n}^{90}(2143)$ with $\pi(k) = 4n$, for all $n \geq 2$ and all k with $2 \leq k \leq 2n$. When $k \geq 3$ we have

$$a_{n,k} = \sum_{j=k-1}^{2n} a_{n-1,j} \qquad \text{(by Lemma 4.16)}$$

$$= \sum_{j=k-1}^{2n} \binom{2n-j-2}{n-j} \qquad \text{(by induction)}$$
$$= \binom{2n-k}{n-k+1},$$

and (i) follows.

Line (ii) is immediate from (i) and Lemma 4.14. □

Our refined enumeration of $S_{4n}^{90}(2143)$ immediately gives us the following refined enumeration of $S_{4n+1}^{90}(2143)$.

Theorem 4.18 *The following hold for all $n \geq 1$.*

(i) *There are exactly $\binom{2n-k}{n-k+1}$ permutations $\pi \in S_{4n+1}^{90}(2143)$ with $\pi(k) = 4n+1$ for $2 \leq k \leq 2n$.*

(ii) *We have*

$$|S_{4n+1}^{90}(2143)| = \binom{2n}{n}. \tag{47}$$

Proof. In view of Theorem 4.17 it is sufficient to show that $\pi \in S_{4n}^{90}$ avoids 2143 if and only if $\pi^u \in S_{4n+1}^{90}$ avoids 2143.

($\Longrightarrow$) Suppose the entries in positions $a < b < c < d$ of π^u form a subsequence α of type 2143. If none of a, b, c, d is $2n+1$ then π also contains a subsequence of type 2143. Otherwise we may argue as in the proof of Lemma 4.9 to obtain a counterexample of length at most 13. By inspecting S_{13}^{90} we find that no such counterexample exists.

($\Longleftarrow$) This is immediate, since π is obtained from π^u by removing $2n+1$. □

Modulo the action of D_8 there is just one $\sigma \in S_4$ for which we have not enumerated $S_n^{90}(\sigma)$. We conclude the paper with a conjecture concerning this enumeration, which we have verified by computer for all $n \leq 8$.

Conjecture 4.19 *For all $n \geq 0$ we have*

$$|S_{4n+1}^{90}(2413)| = |S_{4n}^{90}(2413)| = d_{n+1}, \tag{48}$$

where $d_0 = 1$ and

$$d_n = d_{n-1} + \sum_{k=1}^{n} 2^k C_{k-1} d_{n-k}$$

for all $n \geq 1$. Here C_n is the nth Catalan number.

It is routine to show that for all $n \geq 0$, d_n is the number of lattice paths from $(0,0)$ to (n,n) consisting entirely of North $(0,1)$, East $(0,1)$, and diagonal $(1,1)$ steps, which never pass below the line $y = x$, which only have diagonal steps on the line $y = x$, and in which each North step is colored with one of two colors.

Conjecture 4.19 completes our analysis of the H_3-Wilf-equivalence classes for permutations in S_4. In the table below we summarize the H_3-Wilf-equivalence classes for which $|S_n^{90}(R)| \neq 0$.

σ	$\lvert S_n^{90}(\sigma)\rvert$
1342	Theorem 4.8
1324	Equations (44) and (45)
2143	Equations (46) and (47)
2413	Equation (48)

H_3-Wilf-equivalence classes for elements of S_4

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