Introducing 05A06: Patterns in Permutations and Words

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Carleton College
September 20, 2014
The Case

There are connections with many other areas.
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There are already numerous cool results.
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We’ve answered some deep questions.
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Surprising and exciting new ideas and approaches surface regularly.

There’s room for all, from undergraduates to wily veterans.
Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$. 
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Example

3561274 contains 9 occurrences of 21.
(inversions)
Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$.

Example

3561274 contains 12 occurrences of 12. (coinversions)
The Definition

Definition
Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An \textit{occurrence} of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$.

Example

3561274 contains 7 occurrences of 312.
The Definition

**Definition**

Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$.

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3561274 contains 7 occurrences of 312.
Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$. 

![Diagram](image-url)
Definition

Suppose $\pi$ and $\sigma$ are permutations, written in one-line notation. An occurrence of $\sigma$ in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as the entries of $\sigma$. 
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Observation

Every symmetry $f$ of the square is a bijection between occurrences of $\sigma$ in $\pi$ and occurrences of $\sigma^f$ in $\pi^f$. 
Enumeration Questions

$$\sigma[\pi] := \text{number of occurrences of } \sigma \text{ in } \pi$$
Enumeration Questions

\[ \sigma[\pi] := \text{number of occurrences of } \sigma \text{ in } \pi \]

**Theorem (Rodrigues, 1839)**

\[
\sum_{\pi \in S_n} q^{21[\pi]} = 1(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})
\]
Enumeration Questions

\[\sigma[\pi] := \text{number of occurrences of } \sigma \text{ in } \pi\]

**Theorem (Rodrigues, 1839)**

\[
\sum_{\pi \in S_n} q^{2\sigma[\pi]} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})
\]

**Problem**

For each \(\sigma\), find \(\sum_{\pi \in S_n} q^{\sigma[\pi]}\).
Enumeration Questions

σ[π] := number of occurrences of σ in π

Theorem (Rodrigues, 1839)

\[ \sum_{\pi \in S_n} q^{2[\pi]} = 1(1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \]

Ambition

For each σ, find \( \sum_{\pi \in S_n} q^{\sigma[\pi]} \).
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Dream

For each σ, find \( \sum_{\pi \in S_n} q^{\sigma[\pi]} \).
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Opium-Induced Fever Dream

For each \( \sigma \), find \( \sum_{\pi \in S_n} q^{\sigma[\pi]} \).
Definition

We say $\pi$ avoids $\sigma$ whenever $\sigma[\pi] = 0$. 
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$$Av_n(\sigma) = S_n(\sigma) := \text{set of permutations in } S_n \text{ which avoid } \sigma$$
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$Av_n(\sigma) = S_n(\sigma) :=$ set of permutations in $S_n$ which avoid $\sigma$

Question

For each $n$ and each $\sigma$, what is $|Av_n(\sigma)|$?
Definition

We say $\pi$ avoids $\sigma$ whenever $\sigma[\pi] = 0$.

$Av_n(R) = S_n(R) := \text{set of permutations in } S_n \text{ which avoid all } \sigma \in R$

Question

For each $n$ and each $R$, what is $|Av_n(R)|$?
We say patterns $\sigma_1$ and $\sigma_2$ are *Wilf-equivalent* whenever

$$|Av_n(\sigma_1)| = |Av_n(\sigma_2)|$$

for all $n$. 

**Definition**
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We say patterns $\sigma_1$ and $\sigma_2$ are \textit{Wilf-equivalent} whenever

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for all $n$.

Question

Which patterns of each length are Wilf-equivalent?
### Enumerative Results

| $\sigma$ | $|\text{Av}_n(\sigma)|$ | OGF |
|----------|-------------------------|-----|
| 123      | $\frac{1}{n+1} \binom{2n}{n}$ | $\frac{1 - \sqrt{1 - 4x}}{2x}$ |
| 132      |                         |     |
### More Enumerative Results

| $R$          | $|Av_n(R)|$       | OGF                                      |
|--------------|------------------|------------------------------------------|
| 123, 132     | $2^{n-1}$        | $\frac{1 - x}{1 - 2x}$                  |
| 123, 231     | $1 + \binom{n}{2}$ | $\frac{1 - 2x + 2x^2}{(1 - x)^3}$       |
| 123, 321     | 0 for $n \geq 5$ | $1 + x + 2x^2 + 4x^3 + 4x^4$            |
| 123, 132, 213| $F_{n+1}$        | $\frac{1}{1 - x - x^2}$                 |
| 123, 132, 231| $n$              | $\frac{1}{(1 - x)^2}$                   |
### Even More Enumerative Results

| \( R \) | \( |Av_n(R)| \) | OGF |
|---|---|---|
| 123, 3412 | \( 2^{n+1} - \binom{n+1}{3} - 2n - 1 \) | \( \frac{1 - 5x + 10x^2 - 9x^3 + 4x^4}{(1 - 2x)(1 - x)^4} \) |
| 132, 4231 | \( 1 + (n - 1)2^{n-2} \) | \( \frac{1 - 4x + 5x^2 - x^3}{(1 - 2x)^2(1 - x)} \) |
| 123, 2143 | \( F_{2n} \) | \( \frac{1 - 2x}{1 - 3x + x^2} \) |
### Still More Enumerative Results

| $R$ | $|Av_n(R)|$ | OGF |
|-----|--------------|-----|
| 2143, 3412 | $(2n) - \sum_{m=0}^{n-1} 2^{n-m-1} \binom{2m}{m}$ | $\frac{1 - 3x}{(1 - 2x) \sqrt{1 - 4x}}$ |
| 1234, 3214 | | |
| 4123, 3214 | $4^{n-1} + 2$ | $\frac{x(1 - 3x)}{(1 - x)(1 - 4x)}$ |
| 2341, 2143 | | |
| 1234, 2143 | | |
| 1324, 2143 | | $\frac{1 - 5x + 3x^2 + x^2 \sqrt{1 - 4x}}{1 - 6x + 8x^2 - 4x^3}$ |
| 1342, 2431 | | |
| 1342, 2341 | | |
| 1342, 2314 | | |
| 1324, 2413 | | |
| 2413, 3142 | | |
| 1234, 2134 | | |
| 1324, 2314 | | |
| 3124, 3124 | $r_{n-1} = \sum_{d=0}^{n} C_{n-d} \binom{2n-d}{d}$ | $\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}$ |
| 3142, 3214 | | |
| 3412, 3421 | | |
| 1324, 2134 | | |
| 3124, 2314 | | |
| 2134, 3124 | | |
### Some Open Enumerative Problems

| $R$             | $|\text{Av}_n(R)|$ for $n = 5, 6, 7, 8, 9, 10$ |
|-----------------|-----------------------------------|
| 1234, 3412      | 86, 333, 1235, 4339, 14443, 45770 |
| 1243, 4231      | 86, 335, 1266, 4598, 16016, 53579 |
| 1324, 3412      | 86, 335, 1271, 4680, 16766, 58656 |
| 1324, 4231      | 86, 336, 1282, 4758, 17234, 61242 |
| 1243, 3412      | 86, 337, 1295, 4854, 17760, 63594 |
| 1324, 2341      | 87, 352, 1428, 5768, 23156, 92416 |
| 1342, 4123      | 87, 352, 1434, 5861, 24019, 98677 |
| 1243, 2134      | 87, 354, 1459, 6056, 25252, 105632 |
| 1243, 2431      | 88, 363, 1507, 6241, 25721, 105485 |
| 1324, 2431      | 88, 363, 1508, 6255, 25842, 106327 |
| 1243, 2341      | 88, 365, 1540, 6568, 28269, 122752 |
| 1342, 3412      | 88, 366, 1556, 6720, 29396, 129996 |
| 1243, 2413      | 88, 367, 1568, 6810, 29943, 132958 |
| 1243, 3124      | 88, 367, 1571, 6861, 30468, 137229 |
| 1234, 2341      | 89, 376, 1611, 6901, 29375, 123996 |
| 1342, 2413      | 89, 379, 1664, 7460, 33977, 156727 |
| 1324, 1432      | 89, 380, 1677, 7566, 34676, 160808 |
| 1234, 1342      | 89, 380, 1678, 7584, 34875, 162560 |
| 1432, 2143      | 89, 381, 1696, 7781, 36572, 175277 |
| 1243, 1432      | 89, 382, 1711, 7922, 37663, 182936 |
| 2143, 2413      | 90, 395, 1823, 8741, 43193, 218704 |
### Just A Couple More Enumerative Results

| $\sigma$ | $|Av_n(\sigma)|$ |
|----------|----------------|
| 1234     | $(n+1)(n+2)$  |
| 1243     | $rac{1}{(n+1)^2(n+2)} \sum_{j=0}^{n} (2j)(n+1)(n+2)$ |
| 2143     | $(-1)^{n-1} \frac{7n^2 - 3n - 2}{2} + 3 \sum_{j=2}^{n} \frac{(2j-4)!}{j!(j-2)!} \binom{n-j+2}{2}(-1)^{n-j}2^{j+1}$ |
| 3214     | Unknown beyond $n = 36$ |
“Not even God knows $|\text{Av}_{1000}(1324)|$."

Doron Zeilberger
“Not even God knows $|Av_{1000} (1324)|$.”
Doron Zeilberger

“I’m not sure how good Zeilberger’s God is at math,

Einar Steingrímsson
“Not even God knows $|Av_{1000}(1324)|$.”
Doron Zeilberger

“I’m not sure how good Zeilberger’s God is at math, but I believe that some humans will find this number in the not so distant future.”
Einar Steingrímsson
The Stanley-Wilf Conjecture

**Theorem**

For all $\sigma \in S_3$,

$$\lim_{n \to \infty} n^{\sqrt{\left| Av_n(\sigma) \right|}} = 4.$$
The Stanley-Wilf Conjecture

**Theorem**

For all $\sigma \in S_3$, 

$$\lim_{n \to \infty} n^{\frac{n}{\sqrt{|Av_n(\sigma)|}}} = 4.$$ 

**Wilf’s First Question, \sim 1980**

Is 

$$|Av_n(\sigma)| \leq (|\sigma| + 1)^n$$ 

for all $n$?
The Stanley-Wilf Conjecture

**Theorem**

*For all $\sigma \in S_3$,*

$$\lim_{n \to \infty} n \sqrt{|Av_n(\sigma)|} = 4.$$  

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Theorem

For all $\sigma \in S_3$,

$$\lim_{n \to \infty} n^{\frac{1}{n}} |Av_n(\sigma)| = 4.$$

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$$|Av_n(\sigma)| \leq (|\sigma| + 1)^n$$

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Theorem

For all $\sigma \in S_3$,

$$\lim_{n \to \infty} \sqrt[n]{|Av_n(\sigma)|} = 4.$$ 

Wilf’s First Question, $\sim 1980$

Is

$$|Av_n(\sigma)| \leq (|\sigma| + 1)^n$$

for all $n$?

Theorem (Regev, 1981)

$$\lim_{n \to \infty} \sqrt[n]{|Av_n(12\cdots k)|} = (k - 1)^2$$
The Stanley-Wilf Conjecture

Stanley’s Question, ~ 1980

Is

$$\lim_{n \to \infty} \sqrt[n]{|Av_n(\sigma)|} = (|\sigma| - 1)^2$$

for all $\sigma$?
The Stanley-Wilf Conjecture

Stanley’s Question, ∼ 1980

Is

$$\lim_{n \to \infty} \sqrt[n]{|Av_n(\sigma)|} = (|\sigma| - 1)^2$$

for all $\sigma$?
The Stanley-Wilf Conjecture

Stanley’s Question, \(\sim 1980\)

Is

\[
\lim_{n \to \infty} n \sqrt{|\text{Av}_n(\sigma)|} = (|\sigma| - 1)^2
\]

for all \(\sigma\)?

Wilf’s Next Question

Does there exist, for each \(\sigma\), a constant \(c(\sigma)\) with

\[
\lim_{n \to \infty} n \sqrt{|\text{Av}_n(\sigma)|} = c(\sigma)
\]
The Stanley-Wilf Conjecture

The Stanley-Wilf Upper Bound Conjecture

For every $\sigma$ there is a constant $c(\sigma)$ such that

$$|Av_n(\sigma)| \leq c(\sigma)^n.$$
The Stanley-Wilf Conjecture

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For every $\sigma$ there is a constant $c(\sigma)$ such that

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For every $\sigma$ there is a constant $c(\sigma)$ such that

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For every $\sigma$ there is a constant $c(\sigma)$ such that

$$\lim_{n \to \infty} n^{\frac{1}{\sqrt{n}}} |Av_n(\sigma)| = c(\sigma).$$ 

Limit $\Rightarrow$ Upper Bound: Clear
The Stanley-Wilf Upper Bound Conjecture
For every $\sigma$ there is a constant $c(\sigma)$ such that
\[ |Av_n(\sigma)| \leq c(\sigma)^n. \]

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For every $\sigma$ there is a constant $c(\sigma)$ such that
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Limit $\Rightarrow$ Upper Bound: Clear

Upper Bound $\Rightarrow$ Limit: Arratia 1999
Interlude: Other Notions of Containment

Generalized = Consecutive = Vincular

Example: 25314 contains 2413 but avoids 241.
Interlude: Other Notions of Containment

Generalized $=$ Consecutive $=$ Vincular

$2413$

$2 - 41 - 3$
Interlude: Other Notions of Containment

Generalized = Consecutive = Vincular

2413

2 − 41 − 3
Generalized = Consecutive = Vincular

Example

25314 contains 2413 but avoids 2413.
Bivincular
Bivincular

$\bar{2314}$
Bivincular

Example

315246 contains 2314 but avoids 2314.
Convention: Matrices use only entries 0 and 1.
The Füredi-Hajnal Conjecture

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Definition

A matrix $M$ contains a matrix $C$ whenever $M$ has a submatrix $M_{sub}$ of $C$'s dimensions such that $M_{sub}$ has a 1 in every place $C$ has a 1.
Convention: Matrices use only entries 0 and 1.

Definition

A matrix $M$ contains a matrix $C$ whenever $M$ has a submatrix $M_{\text{sub}}$ of $C$’s dimensions such that $M_{\text{sub}}$ has a 1 in every place $C$ has a 1.

Example

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]
contains
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
The Füredi-Hajnal Conjecture

The Füredi-Hajnal Question, 1992

Given a matrix $C$, how many 1s can an $n \times n$ matrix $M$ contain before it must contain $C$?

Theorem (Klazar, 2001)
Füredi-Hajnal $\Rightarrow$ Stanley-Wilf
The Füredi-Hajnal Conjecture

The Füredi-Hajnal Question, 1992

Given a matrix $C$, how many 1s can an $n \times n$ matrix $M$ contain before it must contain $C$?

The Füredi-Hajnal Conjecture

If $C$ is a permutation matrix then there is a number $c(C)$ such that if an $n \times n$ matrix $M$ has at least $c(C)n$ entries equal to 1, then $M$ contains $C$. 

Theorem (Klazar, 2001)

$\text{Füredi-Hajnal} \implies \text{Stanley-Wilf}$
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\[ \text{Füredi-Hajnal} \Rightarrow \text{Stanley-Wilf} \]
The Marcus-Tardos Theorem

Adam Marcus starts his Fulbright in Hungary, working with Gábor Tardos.

Later in 2003, Marcus and Tardos prove the Füredi-Hajnal conjecture.

Weeks later, Marcus and Tardos learn about the Stanley-Wilf conjecture.
The Marcus-Tardos Theorem

Fall 2003  Adam Marcus starts his Fulbright in Hungary, working with Gábor Tardos
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How Long Did It Take to Prove the Stanley-Wilf Conjecture?

Richard Stanley before
How Long Did It Take to Prove the Stanley-Wilf Conjecture?

Richard Stanley before

Richard Stanley after
Definition

For each \( \sigma \),

\[
L(\sigma) := \lim_{n \to \infty} \sqrt[n]{|Av_n(\sigma)|}.
\]
Growth Rates

Definition

For each $\sigma$, $L(\sigma) := \lim_{n \to \infty} \sqrt[\sigma]{|Av_n(\sigma)|}$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$L(\sigma)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>4</td>
</tr>
<tr>
<td>132</td>
<td></td>
</tr>
<tr>
<td>1234</td>
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<td>3214</td>
<td></td>
</tr>
<tr>
<td>1342</td>
<td></td>
</tr>
<tr>
<td>2413</td>
<td>8</td>
</tr>
<tr>
<td>1324</td>
<td></td>
</tr>
<tr>
<td>$12 \cdots k$</td>
<td>$(k - 1)^2$</td>
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Growth Rates

Definition

For each $\sigma$, $L(\sigma) := \lim_{n \to \infty} \sqrt{n} |A_{\nu_n}(\sigma)|$.

Theorem (Bevan, 2014)

$L(1324) \geq 9.81$

\begin{tabular}{|c|c|}
\hline
$\sigma$ & $L(\sigma)$ \\
\hline
123 & 4 \\
132 & 9 \\
1234 & 8 \\
1243 & 8 \\
2143 & 8 \\
3214 & 8 \\
1342 & 8 \\
2413 & 8 \\
1324 & 8 \\
12 \cdots k & $(k - 1)^2$ \\
\hline
\end{tabular}
### Definition
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### Theorem (Bevan, 2014)

$L(1324) \geq 9.81$

### Theorem (Bóna, 2013)

$L(1324) \leq 13.738$

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Conjecture (Claesson, Jelínek, Steingrímsson, 2012)

For any $\sigma \neq 12 \cdots k$, and any $j \geq 0$, the number of $\sigma$-avoiders with $j$ inversions is a nondecreasing function of length.
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For any $\sigma \neq 12 \cdots k$, and any $j \geq 0$, the number of $\sigma$-avoiders with $j$ inversions is a nondecreasing function of length.

132-avoiders with exactly 2 inversions

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>number</td>
<td>0</td>
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For any $\sigma \neq 12 \cdots k$, and any $j \geq 0$, the number of $\sigma$-avoiders with $j$ inversions is a nondecreasing function of length.

Theorem (Claesson, Jelínek, Steingrímsson, 2012)

If the CJS conjecture holds for $\sigma = 1324$, then

$$L(1324) < 13.001954.$$
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For any $\sigma \neq 12 \cdots k$, and any $j \geq 0$, the number of $\sigma$-avoiders with $j$ inversions is a nondecreasing function of length.

Theorem (Claesson, Jelínek, Steingrímsson, 2012)

If the CJS conjecture holds for $\sigma = 1324$, then

$$L(1324) < e^{\pi \sqrt{2/3}} \approx 13.001954.$$
The Conway-Guttmann Estimate

Conjecture (Conway and Guttmann, 2014)

There are constants $B$, $\mu$, $\mu_1$, and $g$ such that

$$|Av_n(1324)| \sim B \mu^n \sqrt{n} \mu_1 n^g.$$

$\mu = 1.60 \pm 0.01$

$\mu_1 = 0.0398 \pm 0.001$

$g = -1.1 \pm 0.2$

$B = 9.5 \pm 1.0$
Conjecture (Conway and Guttmann, 2014)

There are constants $B$, $\mu$, $\mu_1$, and $g$ such that

$$|\text{Av}_n(1324)| \sim B \mu^n \mu_1^{\sqrt{n}} n^g.$$ 

$$\mu = 11.60 \pm 0.01$$
$$\mu_1 = 0.0398 \pm 0.001$$
$$g = -1.1 \pm 0.2$$
$$B = 9.5 \pm 1.0$$
The Dukes-Parton-West Permutation Patterns Game

- Fix a permutation $\sigma$.
- Players take turns placing stones on grid points.
The Dukes-Parton-West Permutation Patterns Game

- Fix a permutation $\sigma$.
- Players take turns placing stones on grid points.
- No two stones may be in the same row or column.
The Dukes-Parton-West Permutation Patterns Game

Fix a permutation $\sigma$.

Players take turns placing stones on grid points.

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No occurrence of $\sigma$ allowed.
The Dukes-Parton-West Permutation Patterns Game

- Fix a permutation $\sigma$.

- Players take turns placing stones on grid points.

- No two stones may be in the same row or column.

- No occurrence of $\sigma$ allowed.

- Last player to move wins.
Would You Like to Play a Game?

\[ \sigma = 21 \]

Is it better to play first or second?
What If Your Opponent Goes First, But Is Confused?

$\sigma = 21$

Where should you play?
#### A More Complicated Pattern

If \( \sigma = 321 \), should you play first or second?

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Open Problem

Find the general pattern.

Eric S. Egge (Carleton College)
A More Complicated Pattern

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Eric S. Egge (Carleton College) 05A06: Patterns in Permutations and Words September 20, 2014 31 / 34
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Open Problem
Find the general pattern.
Where to Learn More

COMBINATORICS OF PERMUTATIONS
Second Edition
Miklós Bóna

Patterns in Permutations and Words
Sergey Kitaev
The End

Thank You!