Pattern-Avoiding Permutations and Lattice Paths:
Old Connections and New Links

Eric S. Egge

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August 3, 2012
Definition

\pi, \sigma \text{ are permutations.}

\pi \text{ avoids } \sigma \text{ whenever } \pi \text{ has no subsequence with same length and relative order as } \sigma.

Example

6152347 avoids 231 but not 213.
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The diagram of 6152347.
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Notation

\( Av(\sigma) := \text{set of all permutations which avoid } \sigma. \)

\( Av_n(\sigma) = Av(\sigma) \cap S_n \)
Counting Pattern-Avoiding Permutations

\[ |\text{Av}_n(132)| = |\text{Av}_n(213)| = |\text{Av}_n(231)| = |\text{Av}_n(312)| \]

\[ |\text{Av}_n(321)| = |\text{Av}_n(123)| \]
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Idea

Rotation of diagrams gives bijections among these sets.
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Idea

Rotation of diagrams gives bijections among these sets.

Theorem

\[ |Av_n(231)| = |Av_n(321)| = C_n = \frac{1}{n+1} \binom{2n}{n} \]
Definition

A *Catalan path* (of length $n$) is a sequence of $n$ North $(0,1)$ steps and $n$ East $(1,0)$ steps which never passes below the line $y = x$. 

Theorem

The number of Catalan paths of length $n$ is $C_n = \frac{1}{n+1} \binom{2n}{n}$. 
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Recursive Structures

Permutations

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Permutations

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Permutations

$\pi_1 \bigoplus \pi_2

\text{Idea}

F(\pi_1 \bigoplus \pi_2) = NF(\pi_2) - EF(\pi_1)$
Recursive Structures

Permutations

\[ \pi_1 \oplus \pi_2 \]

Paths

Idea

\[ F(\pi_1 \oplus \pi_2) = N F(\pi_2) E F(\pi_1) \]

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Permutations and Lattice Paths

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An inversion in a permutation is an occurrence of the pattern 21.
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**Theorem**

\[
\text{inv}(\pi_1 \oplus \pi_2) = \text{inv}(\pi_1) + \text{inv}(\pi_2) + \text{length}(\pi_2)
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Bonus Information: Inversions

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Theorem

$$\text{inv}(\pi) = \text{area}(F(\pi))$$
Definition

For any permutation $\pi$ and number $k$, let $k(\pi)$ be the number of decreasing subsequences of length $k$ in $\pi$. 
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Definition
The *height* $ht(s)$ of an East step $s$ in a Catalan path $\pi$ is the number of area squares below it. The *$k$th area* of $\pi$ is $area_k(\pi) = \sum_{s \in \pi} \binom{ht(s)}{k-1}$.
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Theorem

\[ k(\pi) = \text{area}_k(F(\pi)) \]

and

\[ \sum_{\pi \in \text{Av}(231)} x_1^{1(\pi)} x_2^{2(\pi)} x_3^{3(\pi)} \cdots = \frac{1}{1 - \frac{x_1}{1 - \frac{x_1 x_2}{1 - \frac{x_1 x_2 x_3}{\cdots}}}}. \]
\[ |\text{Av}_n(321)| = C_n \]

41623785 avoids 321.
Theorem
This process produces a Catalan path for any permutation.

Idea
If the $i$th East step is below $y = x$ then the first $i$ buildings are all height $i - 1$ or less.

Theorem
The restriction to $\text{Av}_n(321)$ is a bijection.

Idea
To avoid 321, we must have increasing heights in the canyons.

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The Schröder Case

A Schröder Path
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\[ r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d} \]
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\[ |Av_n(3421, 3412)| = r_{n-1} \]
The Schröder Case

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Theorem

\[ |\text{Av}_n(3421, 3412)| = r_{n-1} \]

Theorem

\[ k(\pi) = \text{area}_k(F(\pi)) \]

and

\[ \sum_{\pi \in \text{Av}(3421, 3412)} x_1^{1(\pi)} x_2^{2(\pi)} x_3^{3(\pi)} \cdots = 1 + \frac{x_1}{1 - x_1 - \frac{x_1 x_2}{1 - x_1 x_2 - \frac{x_1 x_2 x_3}{\cdots}}} \]
Conjecture

\[ |\text{Av}_n(2413, 2143, 415263)| = r_{n-1} \]
Thank You!