3412-Avoiding Involutions, Continued Fractions, and Chebyshev Polynomials

Eric S. Egge

Gettysburg College

July 5, 2004

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Notation

$S_n(\sigma_1, \ldots, \sigma_m) :=$ permutations in S_n which avoid each of $\sigma_1, \ldots, \sigma_m$

$I_n(\sigma_1, \ldots, \sigma_m) :=$ involutions in $S_n(\sigma_1, \ldots, \sigma_m)$ $(\pi^2 = id)$

$$S(\sigma_1, \ldots, \sigma_m) := \bigcup_{n \geq 0} S_n(\sigma_1, \ldots, \sigma_m)$$

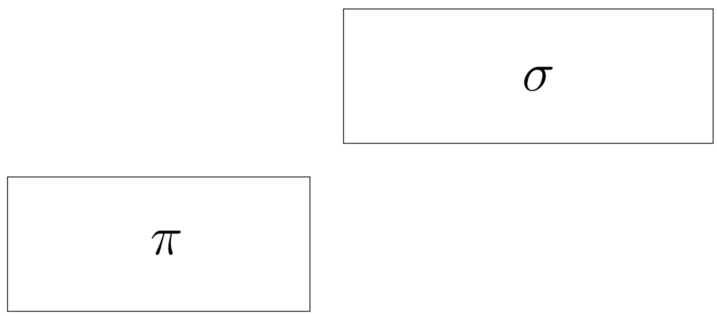
$$I(\sigma_1, \ldots, \sigma_m) := \bigcup_{n \geq 0} I_n(\sigma_1, \ldots, \sigma_m)$$

$$\langle k \rangle := 12 \ldots k$$

$$[k] := k(k-1) \ldots 21$$

Direct Sums

$$(\pi \oplus \sigma)(i) := \begin{cases} \pi(i) & \text{if } 1 \leq i \leq |\pi| \\ |\pi| + \sigma(i - |\pi|) & \text{if } |\pi| + 1 \leq i \leq |\pi| + |\sigma| \end{cases}$$



Example: $[3] \oplus \langle 4 \rangle = 321 \oplus 1234 = 3214567$

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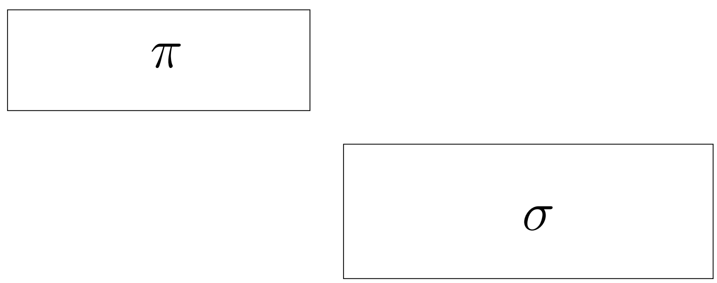
if $\pi \neq \pi_1 \oplus \pi_2$ then π is called $\oplus$-**indecomposable**

Every permutation factors uniquely into $\oplus$-indecomposables

$$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$$

Skew Sums

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i) + |\sigma| & \text{if } 1 \leq i \leq |\pi| \\ \sigma(i - |\pi|) & \text{if } |\pi| + 1 \leq i \leq |\pi| + |\sigma| \end{cases}$$



Example: $\langle 4 \rangle \ominus [2] = 1234 \ominus 21 = 345621$

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if $\pi \neq \pi_1 \ominus \pi_2$ then π is called $\ominus$-**indecomposable**

Every permutation factors uniquely into $\ominus$-indecomposables

$$\pi = \pi_1 \ominus \pi_2 \ominus \cdots \ominus \pi_k$$

$S(132)$ **and Continued Fractions**

Krattenthaler's Bijection:

$S_n(132)$ -------------------> lattice paths $(0,0)$ to (n,n) steps N $(0,1)$ and E $(1,0)$ never below $y=x$

$12\ldots k(\pi)$ -------------------> $\displaystyle\sum_{\substack{s\in\pi\\E}} \binom{ht(s)}{k}$

Applying a result of Flajolet:

$$\sum_{\pi\in S(132)} \prod_{k\geq 1} x_k^{12\ldots k(\pi)} = \cfrac{1}{1-\cfrac{x_1}{1-\cfrac{x_1x_2}{1-\cfrac{x_1x_2^2x_3}{1-\cfrac{x_1x_2^3x_3^3x_4}{1-\cfrac{x_1x_2^4x_3^6x_4^4x_5}{1-\cdots}}}}}}$$

Last line also proved by Robertson, Wilf, and Zeilberger (1999), Mansour and Vainshtein (2000), and Brändén, Claesson, and Steingrímsson (2002).

$S(132)$ Recurrence Relation

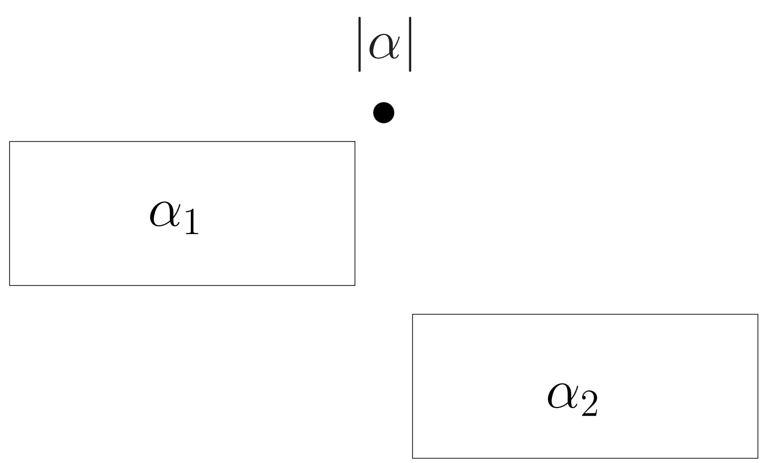
(Mansour and Vainshtein, 2001)

Recursive formula for

$$G_\pi(x) := \sum_{n\geq 0} |S_n(132,\pi)| x^n.$$

$\ominus$-factorization of π is $\pi_1 \ominus \cdots \ominus \pi_k$

$$G_\pi = 1 + x \sum_{i=1}^{k} \left(G_{\overline{\pi_1 \ominus \cdots \ominus \pi_i}} - G_{\overline{\pi_1 \ominus \cdots \ominus \pi_{i-1}}} \right) G_{\pi_i \ominus \cdots \ominus \pi_k}$$



Inclusion-Exclusion allows π to be replaced with any set.

Chebyshev Polynomials of the Second Kind

$$U_n(\cos t) = \frac{\sin((n+1)t)}{\sin t}$$

$$U_{-1}(x) = 0$$

$$U_0(x) = 1$$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

$S(132)$ and Chebyshev Polynomials

(Chow and West, 1999; Krattenthaler, 2001; MV, 2001)

$$G_{\langle k \rangle}(x) = G_{[2]\oplus\langle k-2 \rangle}(x) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)}$$

(MV, 2001)

$$t = \frac{1}{2\sqrt{x}}$$

$$G_{\langle a \rangle \ominus \langle b \rangle \ominus \langle c \rangle}(x) = 2t\frac{U_{a+b}(t)U_{a+c-1}(t)U_{b+c}(t) + U_{b-1}(t)U_b(t)}{U_{a+b}(t)U_{a+c}(t)U_{b+c}(t)}$$

$$G_{\langle a \rangle \ominus \langle b \rangle \ominus \langle c \rangle}(x)$$

$$= 2t\frac{U_{a+b+c}(t)U_{a+b+c-1}(t) + U_{a+b-1}(t)U_{a+c-1}(t)U_{b+c-1}(t)}{U_{a+b}(t)U_{a+c}(t)U_{b+c}(t)}$$

$S(1243, 2143)$ **and Continued Fractions**

Egge and Mansour Bijection:

$S_n(1243, 2143)$ - - - - - - - - - - - - - - -> lattice paths $(0,0)$ to $(n-1, n-1)$ steps N, E, D $(1,1)$ never below $y = x$

$12 \ldots k(\pi)$ - - - - - - - - - - - - - - - - - -> $\delta_{k1} + \sum_{\substack{s \in \pi \\ E \quad D}} \binom{ht(s)}{k-1}$

Applying Flajolet's Result:

$$\sum_{\pi \in S(1243,2143)} \prod_{k \geq 1} x_k^{12\ldots k(\pi)} = 1 + \cfrac{x_1}{1 - x_1 - \cfrac{x_1 x_2}{1 - x_1 x_2 - \cdots}}$$

Bijection given simpler description by Reifegerste (2003).

$S(1243, 2143)$ **Recurrence Relation**

(Egge and Mansour, 2003)

Recursive formula for

$$H_\pi(x) := \sum_{n \geq 0} |S_n(1243, 2143, \pi)| x^n$$

$$H_{\pi \oplus 1} = 1 + xH_{\pi \oplus 1} + (H_\pi - 1)(H_{\pi \oplus 1} - 1)$$

Open Question: Is there a recurrence when π is layered?

$$\pi = \langle l_1, \ldots, l_m \rangle := \langle l_1 \rangle \ominus \cdots \ominus \langle l_m \rangle$$

$S(1243, 2143)$ and Chebyshev Polynomials

(Egge and Mansour, 2003)

$$H_{\langle k \rangle}(x) = 1 + \frac{\sqrt{x}U_{k-2}\left(\frac{1-x}{2\sqrt{x}}\right)}{U_{k-1}\left(\frac{1-x}{2\sqrt{x}}\right)}$$

$$H_{[2]\ominus\langle k-2 \rangle}(x) = 1 + \frac{\sqrt{x}U_{k-2}\left(\frac{1-x}{2\sqrt{x}}\right)}{U_{k-1}\left(\frac{1-x}{2\sqrt{x}}\right)}$$

$$H_{231\ominus\langle k-3 \rangle}(x) = 1 + x\frac{f_{k-1}(x)}{f_k(x)}$$

$$f_2(x) = (x-1)^2$$

$$f_k(x) = (1-2x)^2(\sqrt{x})^{k-3}U_{k-3}\left(\frac{1-x}{2\sqrt{x}}\right)$$
$$-(1-x)^2(\sqrt{x})^{k-2}U_{k-4}\left(\frac{1-x}{2\sqrt{x}}\right)$$

$I(3412)$ **and Lattice Paths**

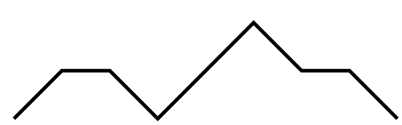
$I_n(3412)$ ----------------> lattice paths $(0,0)$ to $(n,0)$
steps U $(1,1)$, D $(1,-1)$, L $(1,0)$
never below $y=0$

$k\ldots 21(\pi)$ -------------> $\displaystyle\sum_{\substack{s\in\pi\\ L\quad U}} \binom{2ht(s)}{k-1} + \sum_{\substack{s\in\pi\\ D}} \binom{2ht(s)-1}{k-1}$

Bijection:

1. Each entry is one step.
2. Fixed points map to level steps.
3. If $i > \pi(i)$ then ith step is D.
4. If $i < \pi(i)$ then ith step is U.

Example:

32186574 $\longmapsto$ 

$I(3412)$ and Continued Fractions

$$\sum_{\pi \in I(3412)} \prod_{k \geq 1} x_k^{k \ldots 21(\pi)} = \cfrac{1}{1 - x_1 - \cfrac{x_1^2 x_2}{1 - x_1 x_2^2 x_3 - \cdots}}$$

numerator: $\displaystyle\prod_{i=1}^{2n} x_i^{\binom{2n-2}{i-1}+\binom{2n-1}{i-1}}$ denominator: $\displaystyle 1 - \prod_{i=1}^{2n+1} x_i^{\binom{2n}{i-1}}$

statistic	expression	numerator	denominator
inversions	$21(\pi)$	xq^{4n-3}	$1 - xq^{2n}$
l to r maxima	$\sum_{k\geq 1}(-1)^{k-1} k \ldots 1(\pi)$	x^2	$1 - x$
r to l minima	$\sum_{k\geq 1}(-1)^{k-1} k \ldots 1(\pi)$	x^2	$1 - x$
fixed points	$\sum_{k\geq 1}(-2)^{k-1} k \ldots 1(\pi)$	x^2	$1 - xq$

Open Question: Is there a simple interpretation of

$$\sum_{k \geq 1} (-3)^{k-1} k \ldots 1(\pi)?$$

$I(3412)$ **Recurrence Relation**

Define

$$J_\pi(x) := \sum_{n=0}^{\infty} |I_n(3412, \pi)| x^n$$

$\oplus$-factorization of π is $\pi_1 \oplus \cdots \oplus \pi_k$

$$J_\pi = 1 + xJ_\beta + x^2 \sum_{i=1}^{k} \left(J_{\overline{\pi_1 \oplus \cdots \oplus \pi_i}} - J_{\overline{\pi_1 \oplus \cdots \oplus \pi_{i-1}}} \right) J_{\pi_i \oplus \cdots \oplus \pi_k}$$

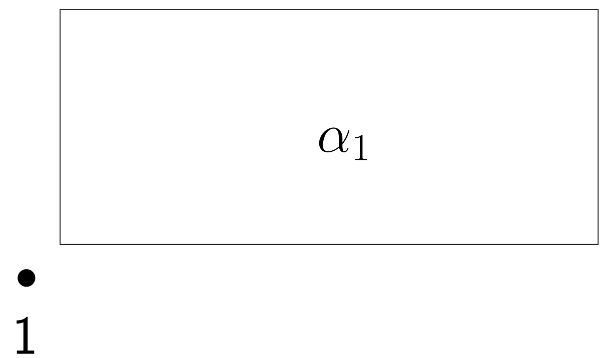
$\beta :=$ remove first entry of π if it's 1.

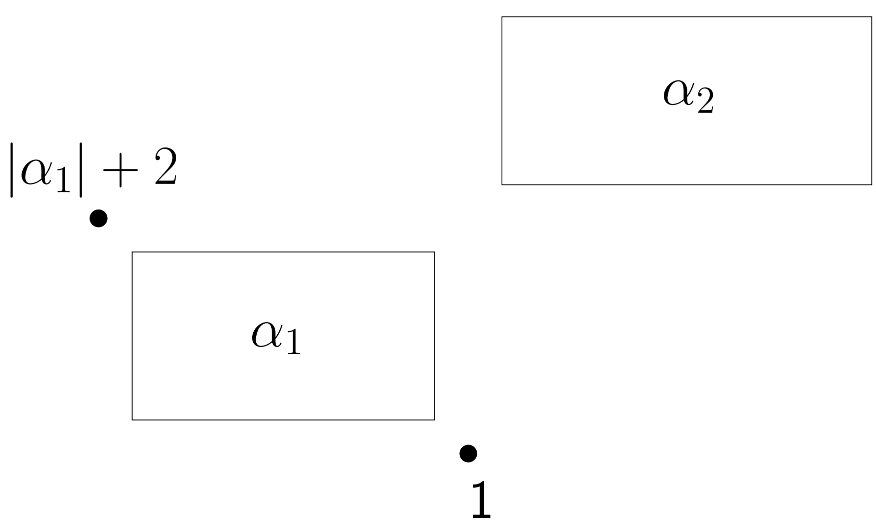
$\overline{\pi} :=$ remove first entry of π if it's largest and remove last entry of π if it's smallest.

Inclusion-Exclusion allows π to be replaced with any set.

$I(3412)$ Recurrence Relation

$$J_\pi = 1 + xJ_\beta + x^2 \sum_{i=1}^{k} \left(J_{\overline{\pi_1\oplus\cdots\oplus\pi_i}} - J_{\overline{\pi_1\oplus\cdots\oplus\pi_{i-1}}}\right) J_{\pi_i\oplus\cdots\oplus\pi_k}$$





$I(3412)$ **Recurrence Relation Example**

$$J_\pi = 1 + xJ_\beta + x^2 \sum_{i=1}^{k} \left(J_{\overline{\pi_1 \oplus \cdots \oplus \pi_i}} - J_{\overline{\pi_1 \oplus \cdots \oplus \pi_{i-1}}} \right) J_{\pi_i \oplus \cdots \oplus \pi_k}$$

$$\pi = 3127564 = 312 \oplus 4231$$

$$J_\pi = 1 + xJ_\pi + x^2(J_{12} - J_\emptyset)J_\pi + x^2(J_\pi - J_{12})J_{4231}$$

$$J_\pi(x) = \frac{1 - x^2 J_{4231} J_{12}}{1 - x - x^2 J_{12} - x^2 J_{4231}}$$

$$J_\pi(x) = \frac{(1-x)(1-2x-x^2)}{1 - 4x + 3x^2 + 2x^3 - x^4}$$

Enumerations

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

$P_0 = 0$
$P_1 = 1$
$P_n = 2P_{n-1} + P_{n-2}$

$$|I_n(3412, 321)| = F_{n+1}$$

$$|I_n(3412, 2143)| = 2^{n-1}$$

$$|I_n(3412, 1243)| = \frac{n}{5}F_{n+2} + \frac{n}{5}F_n - \frac{3}{5}F_n + 1$$

$$|I_n(3412, 3421)| = \frac{P_n + P_{n-1} + 1}{2}$$

$$|I_n(3412, 13254)| = 3 \cdot 2^{n-1} - \frac{2F_{n+3} + nF_{n+2} + F_{n+1} + nF_n}{5}$$

$$|I_n(3412, 54312)| = \frac{1}{2}\left(F_{n+1} + F_{2n-1}\right)$$

$I(3412)$ and Chebyshev Polynomials

$$V_k := U_k\left(\frac{1-x}{2x}\right)$$

$$J_{[2k]}(x) = \frac{V_{k-1}}{xV_k} \qquad J_{[2k-1]}(x) = \frac{V_{k-1}+V_{k-2}}{x\left(V_k+V_{k-1}\right)}$$

$$J_{[k-2]\ominus 231}(x) = J_{[k-2]\ominus\langle 2\rangle}(x) = \frac{V_{k-2}}{xV_{k-1}}$$

$$J_{[k-3]\ominus 213}(x) = J_{[k-3]\ominus 132}(x) = \frac{V_{k-2}+V_{k-3}}{x\left(V_{k-1}+V_{k-2}\right)}$$

$$J_{[k-3]\ominus\langle 3\rangle}(x) = \frac{(1-x+x^3)V_{k-3}+(x-1)xV_{k-4}}{(1-x+x^3)xV_{k-2}+(x-1)x^2V_{k-3}}$$

Layered Permutations and a Conjecture

$$[l_1, \ldots, l_m] := [l_1] \oplus [l_2] \oplus \cdots \oplus [l_m]$$

Theorem (E):

$$J_{[k,l]} = J_{[k+l]}$$

Theorem (E):

$$J_{[2k_1,2k_2,2k_3]} = \frac{V_{k_1+k_2+k_3} V_{k_1+k_2+k_3-1} + V_{k_1+k_2-1} V_{k_1+k_3-1} V_{k_2+k_3-1}}{x V_{k_1+k_2} V_{k_1+k_3} V_{k_2+k_3}}$$

Conjecture: $J_{[l_1,\ldots,l_m]}$ is symmetric in $l_1, \ldots, l_m$.

Proved for $m = 2, 3$ and verified with Maple for $m = 4,\ \ l_i \leq 20$ and $m = 5,\ \ l_i \leq 11$.

A Connection Between $S(132)$ and $I(3412)$

Recall:

$$J_{[2k_1,2k_2,2k_3]} = \frac{V_{k_1+k_2+k_3}V_{k_1+k_2+k_3-1} + V_{k_1+k_2-1}V_{k_1+k_3-1}V_{k_2+k_3-1}}{xV_{k_1+k_2}V_{k_1+k_3}V_{k_2+k_3}}$$

$G_{\langle a,b,c\rangle}(x)$

$$= 2t\frac{U_{a+b+c}(t)U_{a+b+c-1}(t) + U_{a+b-1}(t)U_{a+c-1}(t)U_{b+c-1}(t)}{U_{a+b}(t)U_{a+c}(t)U_{b+c}(t)}$$

Theorem (E):

$$(1-x)J_{[2l_1,\ldots,2l_k]}(x) = G_{\langle l_1,\ldots,l_k\rangle}\left(\frac{x^2}{(1-x)^2}\right)$$

Conjecture: $G_{\langle l_1,\ldots,l_m\rangle}$ is symmetric in $l_1,\ldots,l_m$.

Even and Odd Permutations in $I(3412)$

(Joint work with Toufik Mansour)

Continued Fractions

$$C^{+}(x_1, x_2, \ldots) := \sum_{\pi \in I(3412)} \prod_{k \geq 1} x_k^{k\ldots 21(\pi)}$$

$$C^{-}(x_1, x_2, \ldots) := \sum_{\pi \in I(3412)} (-1)^{sign(\pi)} \prod_{k \geq 1} x_k^{k\ldots 21(\pi)}$$

$$C^{-}(x_1, x_2, \ldots) = C^{+}(x_1, -x_2, x_3, \ldots)$$

C^{-} has a nice continued fraction expansion.

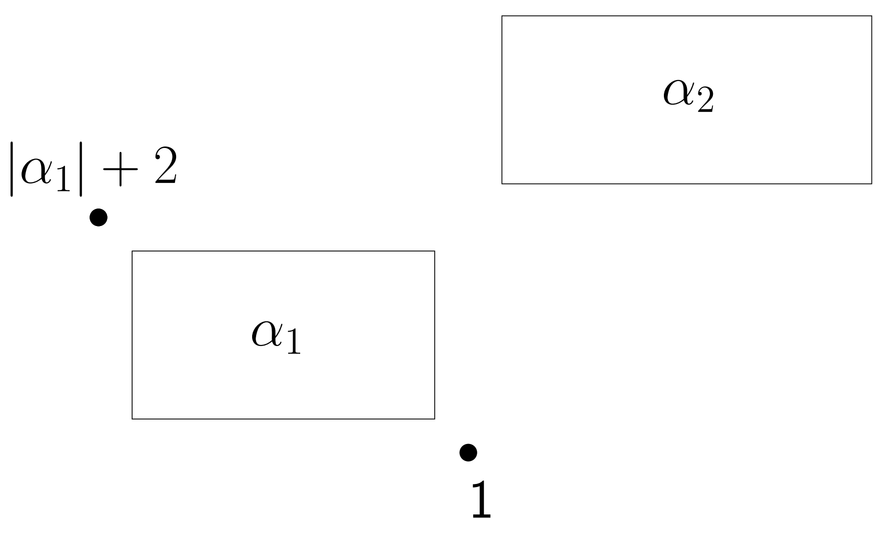
Even and Odd Permutations in $I(3412)$

(Joint work with Toufik Mansour)

Recurrence Relation

$$J^{-}_{\pi}(x) = \sum_{\pi \in I(3412)} (-1)^{sign(\pi)} x^{|\pi|}$$

$$J^{-}_{\pi} = 1 + xJ^{-}_{\beta} - x^2 \sum_{i=1}^{k} \left(J^{-}_{\overline{\pi_1 \oplus \cdots \oplus \pi_i}} - J^{-}_{\overline{\pi_1 \oplus \cdots \oplus \pi_{i-1}}} \right) J^{-}_{\pi_i \oplus \cdots \oplus \pi_k}$$



Even and Odd Permutations in $I(3412)$

(Joint work with Toufik Mansour)

Chebyshev Polynomials I

$$W_n(x) = U_n\left(\frac{1-x}{2ix}\right)$$

$$J^-_{[2k]}(x) = \frac{W_{k-1}}{ixW_k} \qquad J^-_{[2k-1]}(x) = \frac{W_{k-2} - ixW_{k-3}}{ix(W_{k-1} - ixW_{k-2})}$$

$$J^-_{[k-3]\ominus 213}(x) = J^-_{[k-3]\ominus 132}(x) = \frac{W_{k-2} - ixW_{k-3}}{ix(W_{k-1} - ixW_{k-2})}$$

If k and l are not both odd then

$$J^-_{[k,l]}(x) = J^-_{[k+l]}(x).$$

$$J^-_{[2k_1-1,2k_2-1]}(x) = \frac{W_{k_1+k_2-1} - 2iW_{k_1+k_2-2} - W_{k_1+k_2-3}}{ix(W_{k_1+k_2} - 2iW_{k_1+k_2-1} - W_{k_1+k_2-2})}$$

$$J^-_{[2k_1,2k_2,2k_3]} = \frac{W_{k_1+k_2+k_3}W_{k_1+k_2+k_3-1} + W_{k_1+k_2-1}W_{k_1+k_3-1}W_{k_2+k_3-1}}{ixW_{k_1+k_2}W_{k_1+k_3}W_{k_2+k_3}}$$

Even and Odd Permutations in $I(3412)$

(Joint work with Toufik Mansour)

Chebyshev Polynomials II

Conjecture: $J^{-}_{[l_1,\ldots,l_m]}(x)$ is symmetric in $l_1,\ldots,l_m$.

Proved for $m = 2, 3$ and verified with Maple for $m = 4, \;\; l_i \leq 20$ and $m = 5, \;\; l_i \leq 11$.

For More Details

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