Snow Leopard Permutations, Even Knots, Odd Knots, Janus Knots, and Restricted Catalan Paths

Ben Caffrey, Eric Egge*, Greg Michel, Kailee Rubin, Jon Ver Steegh

Carleton College

May 21, 2015
My Students

Ben  Greg  Jon  Kailee
A Problem in Analysis

$f$ and $g$ are functions from $[0, 1]$ into $[0, 1]$ which commute:

$$f(g(t)) = g(f(t)).$$
$f$ and $g$ are functions from $[0, 1]$ into $[0, 1]$ which commute:

\[ f(g(t)) = g(f(t)). \]

**Conjecture (Dyer, 1954)**

$f$ and $g$ must have a common fixed point.
$f$ and $g$ are functions from $[0, 1]$ into $[0, 1]$ which commute:

$$f(g(t)) = g(f(t)).$$

Conjecture (Dyer, 1954)

$f$ and $g$ must have a common fixed point.

Observation (Baxter, 1964)

$f$ and $g$ permute the fixed points of $g \circ f = f \circ g.$
Complete Baxter Permutations

Definition

$\pi$ is a complete Baxter permutation if for all $i$ with $1 \leq i \leq |\pi|$: 
- $\pi(i)$ is even if and only if $i$ is even
Complete Baxter Permutations

Definition

\(\pi\) is a complete Baxter permutation if for all \(i\) with \(1 \leq i \leq |\pi|\):

1. \(\pi(i)\) is even if and only if \(i\) is even
2. if \(\pi(x) = i\), \(\pi(z) = i + 1\), and \(y\) is between \(x\) and \(z\), then \(\pi(y) < i\) if \(i\) is odd and \(\pi(y) > i + 1\) if \(i\) is even

Example

3 2 1 4 13 12 7 8 11 10 9 6 5
Complete Baxter Permutations

Definition

\(\pi\) is a complete Baxter permutation if for all \(i\) with \(1 \leq i \leq |\pi|\):

- \(\pi(i)\) is even if and only if \(i\) is even
- if \(\pi(x) = i, \pi(z) = i + 1\), and \(y\) is between \(x\) and \(z\), then \(\pi(y) < i\) if \(i\) is odd and \(\pi(y) > i + 1\) if \(i\) is even

Example

3 2 1 4 13 12 7 8 11 10 9 6 5
Complete Baxter Permutations

**Definition**

\(\pi\) is a complete Baxter permutation if for all \(i\) with \(1 \leq i \leq |\pi|\):

- \(\pi(i)\) is even if and only if \(i\) is even
- if \(\pi(x) = i\), \(\pi(z) = i + 1\), and \(y\) is between \(x\) and \(z\), then \(\pi(y) < i\) if \(i\) is odd and \(\pi(y) > i + 1\) if \(i\) is even

**Example**

\[
\begin{array}{cccccccc}
13 & 12 & 7 & 8 & 11 & 10 & 9 & 6 \\
3 & 2 & 1 & 4 & & & & \\
3 & 2 & 1 & 4 & & & & \end{array}
\]
Baxter Permutations and anti-Baxter Permutations

**Permutation in the odd entries:**
- Determines a unique complete Baxter permutation
- Commonly called a (reduced) Baxter permutation
- Is characterized by avoiding the generalized patterns 3 – 14 – 2 and 2 – 41 – 3
Baxter Permutations and anti-Baxter Permutations

Permutation in the odd entries:
- Determines a unique complete Baxter permutation
- Commonly called a (reduced) Baxter permutation
- Is characterized by avoiding the generalized patterns 3 − 14 − 2 and 2 − 41 − 3

Permutation in the even entries:
- May not determine a unique complete Baxter permutation
- Has no common name, though sometimes called an anti-Baxter permutation
- Is characterized by avoiding the generalized patterns 3 − 41 − 2 and 2 − 14 − 3
Compatibility

Definition

If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are compatible.

Examples

Each Baxter permutation is compatible with a unique anti-Baxter permutation.
Compatibility

**Definition**
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are *compatible*.

**Examples**
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

\[ 1 \ 4 \ 3 \ 2 \ 5 \]
Compatibility

Definition
If there exists a complete Baxter permutation \( \pi \) such that \( \pi_1 \) and \( \pi_2 \) are the permutations induced on the odd and even entries of \( \pi \), respectively, we say that \( \pi_1 \) and \( \pi_2 \) are compatible.

Examples
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

\[
1 \ 1 \ 4 \ 3 \ 3 \ 2 \ 2 \ 4 \ 5
\]
Compatibility

Definition
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are compatible.

Examples
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

1 2 7 6 5 4 3 8 9
Compatibility

Definition
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are compatible.

Examples
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

1 2 7 6 5 4 3 8 9

Anti-Baxter permutations may be compatible with multiple Baxter permutations.
Compatibility

Definition
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are compatible.

Examples
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

\[1 \ 2 \ 7 \ 6 \ 5 \ 4 \ 3 \ 8 \ 9\]

Anti-Baxter permutations may be compatible with multiple Baxter permutations.

\[1 \ 3 \ 2 \ 4\]
Compatibility

**Definition**
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are *compatible*.

**Examples**
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

$$1\ 2\ 7\ 6\ 5\ 4\ 3\ 8\ 9$$

Anti-Baxter permutations may be compatible with multiple Baxter permutations.

$$1\ 1\ 4\ 3\ 3\ 2\ 2\ 4\ 5$$
$$1\ 1\ 3\ 3\ 4\ 2\ 2\ 4\ 5$$
$$1\ 1\ 4\ 3\ 2\ 2\ 3\ 4\ 5$$
**Compatibility**

**Definition**
If there exists a complete Baxter permutation $\pi$ such that $\pi_1$ and $\pi_2$ are the permutations induced on the odd and even entries of $\pi$, respectively, we say that $\pi_1$ and $\pi_2$ are *compatible*.

**Examples**
Each Baxter permutation is compatible with a unique anti-Baxter permutation.

\[
\begin{align*}
1 & 2 7 6 5 4 3 8 9 \\
\end{align*}
\]

Anti-Baxter permutations may be compatible with multiple Baxter permutations.

\[
\begin{align*}
1 & 2 7 6 5 4 3 8 9 \\
1 & 2 5 6 7 4 3 8 9 \\
1 & 2 7 6 3 4 5 8 9 \\
\end{align*}
\]
The number of Baxter permutations compatible with a given anti-Baxter permutation is a product of Fibonacci numbers.
Aztec Diamond
Aztec Diamond
Aztec Diamond
Aztec Diamond
### Doubly Alternating Baxter Permutations

- Ascents and descents alternate in $\pi$, beginning with an ascent.
- Ascents and descents alternate in $\pi^{-1}$, beginning with an ascent.
- Baxter
DABPs

Doubly Alternating Baxter Permutations
- ascents and descents alternate in $\pi$, beginning with an ascent
- ascents and descents alternate in $\pi^{-1}$, beginning with an ascent
- Baxter

Theorem (Guibert & Linusson, 2000)
The number of DABPs of length $2n$ is $C_n$, the $n^{th}$ Catalan number.
Snow Leopard Permutations

Definition
We call the permutations of length \( n \) which are compatible with the DABPs of length \( n + 1 \) the **snow leopard permutations** (SLPs).

Examples
1
123, 321
12345, 14325, 34521, 54123, 54321

Properties
- anti-Baxter
- identity and reverse identity are always snow leopard
- odd entries in odd positions, even entries in even positions
Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation $\pi$ of length $2n$ is an SLP if and only if there exists an SLP $\sigma$ of length $2n - 1$ such that $\pi = 1 \oplus \sigma^c$. 
Decomposition of SLPs

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation \( \pi \) of length \( 2n \) is an SLP if and only if there exists an SLP \( \sigma \) of length \( 2n - 1 \) such that \( \pi = 1 \oplus \sigma^c \).

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation \( \pi \) is an SLP if and only if there exist SLPs \( \pi_1 \) and \( \pi_2 \) such that \( \pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2 \).
Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation $\pi$ of length $2n$ is an SLP if and only if there exists an SLP $\sigma$ of length $2n - 1$ such that $\pi = 1 \oplus \sigma^c$.

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation $\pi$ is an SLP if and only if there exist SLPs $\pi_1$ and $\pi_2$ such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. 

\[587694321\]
Decomposition of SLPs

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)
A permutation $\pi$ of length $2n$ is an SLP if and only if there exists an SLP $\sigma$ of length $2n - 1$ such that $\pi = 1 \oplus \sigma^c$.

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)
A permutation $\pi$ is an SLP if and only if there exist SLPs $\pi_1$ and $\pi_2$ such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$. 

$123^c$

587694321

$(1 \oplus 123^c \oplus 1) \ominus 1 \ominus 321$
Decomposition of SLPs

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

A permutation $\pi$ is an SLP if and only if there exist SLPs $\pi_1$ and $\pi_2$ such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$.

Theorem (Caffrey, Egge, Michel, Rubin, Ver Steegh)

$SL_n :=$ the set of snow leopard permutations of length $2n - 1$

- $|SL_1| = 1$, $|SL_2| = 2$
- $|SL_{n+1}| = \sum_{j=0}^{n} |SL_j||SL_{n-j}|$
- $|SL_n| = C_n$
Bijection with Catalan paths

<table>
<thead>
<tr>
<th>3</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>7</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
</table>
Bijection with Catalan paths

8  3  6  5  4  7  2  1  0
Bijection with Catalan paths

\[ \begin{array}{ccccccccccc}
8 & 3 & 6 & 5 & 4 & 7 & 2 & 1 & 0 \\
 \downarrow & a & \downarrow & d & \downarrow & d & a & d & d & d & d
\end{array} \]
Bijection with Catalan paths

<table>
<thead>
<tr>
<th>8</th>
<th>3</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>7</th>
<th>2</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>a</td>
<td>d</td>
<td>d</td>
<td>a</td>
<td>d</td>
<td>d</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>N</td>
<td>E</td>
<td>N</td>
<td>N</td>
<td>E</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>
Bijection with Catalan paths

\[ \begin{array}{cccccccc}
8 & 3 & 6 & 5 & 4 & 7 & 2 & 1 & 0 \\
\text{d} & a & d & d & a & d & d & d \\
N & N & N & E & E & E & N & E \\
\end{array} \]
Bijection with Catalan paths

\[
\begin{array}{ccccccccccc}
8 & 3 & 6 & 5 & 4 & 7 & 2 & 1 & 0 \\
\text{d} & \text{a} & \text{d} & \text{d} & \text{a} & \text{d} & \text{d} & \text{d} \\
\text{N} & \text{N} & \text{N} & \text{E} & \text{E} & \text{E} & \text{E} & \text{N} & \text{E} \\
\end{array}
\]
Odd and Even Knots

Definition
We call the permutation induced on the even entries of an SLP $\pi$ an *even knot* ($\text{even}(\pi)$) and the permutation induced on the odd entries an *odd knot* ($\text{odd}(\pi)$).
Odd and Even Knots

Definition
We call the permutation induced on the even entries of an SLP $\pi$ an *even knot* (even($\pi$)) and the permutation induced on the odd entries an *odd knot* (odd($\pi$)).

Examples
Odd knots: $\emptyset$, 1, 12, 21, 123, 231, 312, 321
Even knots: $\emptyset$, 1, 12, 21, 123, 132, 213, 231, 312, 321
Decomposition of Even and Odd Knots

Odd knot $\beta$ decomposition

Even knot $\alpha$ decomposition
What are the odd and even knots counted by?

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>EK_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>$</td>
<td>OK_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>
What are the odd and even knots counted by?

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>E K_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>$</td>
<td>O K_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

**Theorem (Egge, Rubin)**

The odd knots of length $n$ are in bijection with the set of Catalan paths of length $n$ which do not contain NEEN.
What are the odd and even knots counted by?

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>EK_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>17</td>
</tr>
<tr>
<td>$</td>
<td>OK_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
</tr>
</tbody>
</table>

**Theorem (Egge, Rubin)**

The odd knots of length $n$ are in bijection with the set of Catalan paths of length $n$ which do not contain NEEN.

**Theorem (Egge, Rubin)**

The even knots of length $n$ are in bijection with the set of Catalan paths of length $n + 1$ which have no ascent of length exactly 2. (Essentially no ENNE.)
Entangled Knots

Definition

We say an even knot $\alpha$ and an odd knot $\beta$ are entangled whenever there exists an SLP $\pi$ such that $\text{even}(\pi) = \alpha$ and $\text{odd}(\pi) = \beta$. 

Theorem (Egge, Rubin)

The even knots of length $n-1$ entangled with the identity permutation of length $n$ are the 3412-avoiding involutions of length $n-1$.

Theorem (Egge, Rubin)

The odd knots of length $n+1$ entangled with the reverse identity permutation of length $n$ are the complements of the 3412-avoiding involutions of length $n+1$. 

Entangled Knots

Definition
We say an even knot $\alpha$ and an odd knot $\beta$ are entangled whenever there exists an SLP $\pi$ such that $\text{even}(\pi) = \alpha$ and $\text{odd}(\pi) = \beta$.

Theorem (Egge, Rubin)
The even knots of length $n - 1$ entangled with the identity permutation of length $n$ are the 3412-avoiding involutions of length $n - 1$. 
Theorem (Egge, Rubin)

The even knots of length $n - 1$ entangled with the identity permutation of length $n$ are the 3412-avoiding involutions of length $n - 1$.

Theorem (Egge, Rubin)

The odd knots of length $n + 1$ entangled with the reverse identity permutation of length $n$ are the complements of the 3412-avoiding involutions of length $n + 1$. 
Motzkin Numbers

\( M_n \) is the number of lattice paths from \((0,0)\) to \((n,0)\) using only up \((1,1)\), level \((1,0)\), and down \((1,-1)\) steps.
Motzkin Numbers

$M_n$ is the number of lattice paths from $(0, 0)$ to $(n, 0)$ using only up $(1, 1)$, level $(1, 0)$, and down $(1, -1)$ steps.
Motzkin Numbers

\( M_n \) is the number of lattice paths from \((0, 0)\) to \((n, 0)\) using only up \((1, 1)\), level \((1, 0)\), and down \((1, -1)\) steps.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_n )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>21</td>
<td>51</td>
<td>127</td>
<td>323</td>
<td>835</td>
</tr>
</tbody>
</table>
Corollary (Egge, Rubin)

The number of even knots of length \( n - 1 \) entangled with the identity permutation of length \( n \) is \( M_{n-1} \), where \( M_n \) is the \( n^{th} \) Motzkin number.

Corollary (Egge, Rubin)

The number of odd knots of length \( n + 1 \) entangled with the reverse identity permutation of length \( n \) is \( M_{n+1} \).
Corollary (Egge, Rubin)

The number of even knots of length $n - 1$ entangled with the identity permutation of length $n$ is $M_{n-1}$, where $M_n$ is the $n^{th}$ Motzkin number.

Corollary (Egge, Rubin)

The number of odd knots of length $n + 1$ entangled with the reverse identity permutation of length $n$ is $M_{n+1}$.

Conjecture

For each even (resp. odd) knot, the number of entangled odd (resp. even) knots is a product of Motzkin numbers.
Janus Knots

<table>
<thead>
<tr>
<th>Odd Knots</th>
<th>Even Knots</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>123</td>
<td>123</td>
</tr>
<tr>
<td>231</td>
<td>132</td>
</tr>
<tr>
<td>312</td>
<td>213</td>
</tr>
<tr>
<td>321</td>
<td>231</td>
</tr>
<tr>
<td>1234</td>
<td>312</td>
</tr>
<tr>
<td>1324</td>
<td>321</td>
</tr>
<tr>
<td>2341</td>
<td>1234</td>
</tr>
<tr>
<td>3412</td>
<td>1243</td>
</tr>
<tr>
<td>3421</td>
<td>1324</td>
</tr>
<tr>
<td>4123</td>
<td>1432</td>
</tr>
<tr>
<td>4231</td>
<td>2134</td>
</tr>
<tr>
<td>4312</td>
<td>2143</td>
</tr>
<tr>
<td>4321</td>
<td>2341</td>
</tr>
<tr>
<td>12345</td>
<td>2431</td>
</tr>
<tr>
<td>12435</td>
<td>3214</td>
</tr>
</tbody>
</table>
# Janus Knots

## Odd Knots

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>21</td>
<td>123</td>
<td>231</td>
</tr>
<tr>
<td>321</td>
<td>1234</td>
<td>321</td>
<td>12345</td>
<td>12435</td>
</tr>
</tbody>
</table>

## Even Knots

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>21</td>
<td>123</td>
<td>231</td>
</tr>
<tr>
<td>321</td>
<td>1234</td>
<td>321</td>
<td>12345</td>
<td>12435</td>
</tr>
</tbody>
</table>

**Definition**

A Janus knot is a permutation which is both an even knot and an odd knot.
# Janus Knots

## Odd Knots

1

12

21

123

231

321

321

1234

1324

2341

3412

3421

4123

4231

4312

4321

12345

12435

## Even Knots

1

12

21

123

132

213

231

312

321

1234

1243

1324

1432

2134

2143

2314

2341

3214

3421

4123

4231

4312

4321

12345

12435

---

**Definition**

A *janus knot* is a permutation which is both an even knot and an odd knot.
### Janus Knots

#### Odd Knots

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>J_n</td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>17</td>
<td>37</td>
<td>82</td>
</tr>
</tbody>
</table>

#### Even Knots

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>J_n</td>
<td>$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>17</td>
<td>37</td>
<td>82</td>
</tr>
</tbody>
</table>

### Definition

A *janus knot* is a permutation which is both an even knot and an odd knot.
Janus Knots and Motzkin Paths

A004148 Generalized Catalan numbers: $a(n+1) = a(n) + \sum(k=1..n-1, a(k) \cdot a(n-1-k))$.
(Formerly M1141)

1, 1, 1, 2, 4, 8, 17, 37, 82, 185, 423, 978, 2283, 5373, 12735, 30372, 72832, 175502, 424748, 1032004, 2516347, 6155441, 15101701, 37150472, 91618049, 226460893, 560954047, 1392251012, 3461824644, 8622571758, 21511212261, 53745962199, 134474581374 (list; graph; refs; listen; history; text; internal format)

OFFSET

4

COMMENTS

Arises in enumerating secondary structures of RNA molecules. The 17 structures with 6 nucleotides are shown in the illustration (after Waterman, 1978).

Hankel transform is period 8 sequence [1, 1, 1, 0, -1, -1, -1, 0, ...].

Enumerates peak-less Motzkin paths of length n. Example: a(5) = 8 because we have HHHHH, HHUHD, HUHHD, UHHHD, UHHDH, UHHHD, UUHDD, where U = (1, 1), D = (1, -1) and H = (0, 0). - Emeric Deutsch, Nov 19 2003

Number of Dyck paths of semilength $n-1$ with no UUU's and no DDD's, where $U = (1, 1)$ and $D = (1, -1)$ (n>0) - Emeric Deutsch, Nov 19 2003

For n>1, a(n) = number of dissections of an (n+2)-gon with strictly disjoint diagonals and no diagonal incident with the base. (One side of the (n+2)-gon is designated the base.) - David Callan, Mar 23 2004

For n>2, a(n-2) = number of UU-free Motzkin n-paths = number of DU-free Motzkin n-paths. - David Callan, Jul 15 2004

a(n) = number of UU-free Motzkin n-paths containing no low peaks (A low peak is a UD pair at ground level, i.e. whose removal would create a pair of Motzkin paths). For n>1, a(n) = number of DU-free Motzkin (n-1)-paths = number of DU-free Motzkin (n-1)-paths. a(n) is asymptotically ~ c n^(-3/2) (1 + phi)^n with c = 1.1043... and phi = (1+sqrt(5))/2. - David Callan, Jul 15 2004.

In closed form, c = sqrt((30+14*sqrt(5))/(4*sqrt(Pi))) = 1.10435547309692849... - Václav Kotesovec, Sep 11 2013

a(n) = number of Dyck (n+1)-paths with all pyramid sizes >= 2. A pyramid is a maximal subpath of the form k upsteps immediately followed by k downsteps and its size is k. - David Callan, Oct 24 2004

a(n) = number of Dyck paths of semilength n+1 with no small pyramids (n>1). A pyramid is a maximal sequence of the form k Us followed by k Ds with k>=1. A small pyramid is one with k=1. For example, a[4]=4 counts the following Dyck 5-paths (pyramids denoted by lowercase letters and separated by a vertical bar): uuuuudddd, uudd|uuddD, uudd|uuudd, uuudd|uudd. - David Callan, Oct 25 2004

From Emeric Deutsch, Jan 08 2006: (Start)
a(n) = number of Motzkin paths of length n-1 with no peaks at level >= 1.
Theorem (Egge, Rubin)

There is a natural bijection between the set of janus knots of length $n$ and the set of peakless Motzkin paths of length $n + 1$.
Theorem (Egge, Rubin)

There is a natural bijection between the set of janus knots of length $n$ and the set of peakless Motzkin paths of length $n + 1$. 
Thank you!