Kepler Walls

Anna Duane
Department of Mathematics
Carleton College
Northfield, MN 55057 USA
anna.duane@alumni.carleton.edu

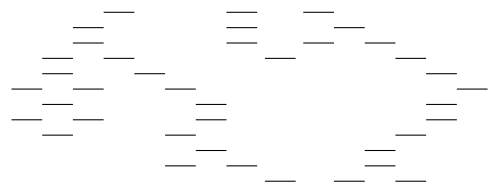
Eric S. Egge
Department of Mathematics
Carleton College
Northfield, MN 55057 USA
eggee@member.ams.org

Abstract

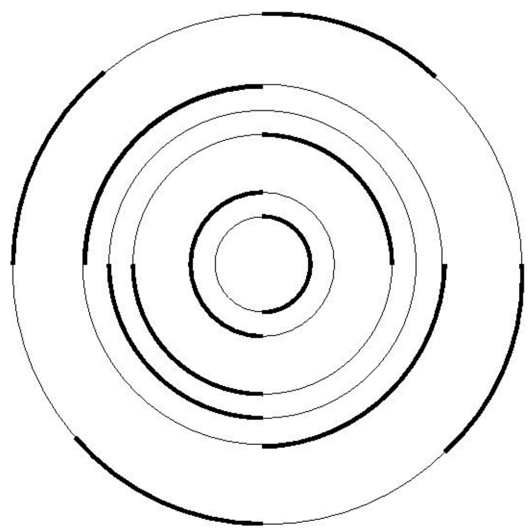
Motivated by Viennot's Kepler towers, we introduce and study a family of related objects called Kepler walls. Roughly speaking, a Kepler wall consists of rows of bricks in which no two bricks are adjacent and every brick not in the top row is supported by a brick in the row above. We show that for a variety of top rows the number of Kepler walls with n bricks is given by a binomial coefficient, by giving a constructive bijection to an appropriate set of lattice paths. We also use Chebyshev polynomials of the second kind and the transfer matrix method to obtain generating functions for various families of Kepler walls of restricted width. In the process we find connections with Fibonacci numbers, order-consecutive partitions, and walks in a certain path.

1 Introduction

Like many walls, Kepler walls are built of *bricks*, which in this case are line segments of length 1. Bricks in a Kepler wall are arranged in rows in which the distance between the midpoints of any two distinct bricks is an integer greater than one. Thus, each row is divided into *brick spaces* of length 1, each of which may or may not contain a brick, with the restriction that two adjacent brick spaces cannot both contain bricks. A *Kepler wall* is a stack of rows of bricks in which brick spaces are aligned vertically and every brick not in the top row has a brick directly above, directly above and one space to its right, or directly above and once space to its left. We give an example of a Kepler wall below.



Our interest in Kepler walls is due in part to their relationship with Kepler towers, which were first introduced by Viennot and have recently been studied by Knuth [2]. Roughly speaking, a *Kepler tower* is a sequence of circular Kepler walls; in this context we call each wall a *turret* and each row a *ring*. The innermost ring in a turret in a Kepler tower corresponds to the top row of a Kepler wall, and the turrets of a Kepler tower are aligned so that each ring has a topmost brick space. By definition each ring in the kth turret of a Kepler tower has 2^k brick spaces, the innermost ring in the kth turret contains 2^{k-1} bricks, and one of these bricks occupies the topmost brick space in the ring. As the example below illustrates, the rings in a Kepler tower are generally represented as concentric circles, with the rings of the first turret having the smallest radii. Knuth has given [2] bijections between Kepler towers and several families of objects counted by the Catalan numbers, so the number of Kepler towers with exactly n bricks is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.



In this paper, we will enumerate several families of Kepler walls, beginning with the Kepler half-walls, which we define next.

Definition 1.1 *A* right Kepler half-wall *is a Kepler wall with one top row brick space and no bricks to the left of the root brick, which is the brick in the top row. A* left Kepler half-wall *is the reflection of a right Kepler half-wall over a vertical line. We say a right (resp. left) Kepler half-wall is* irreducible *whenever the only brick in the leftmost (resp. rightmost) column is the root.*

In Section 2 we show that the number of irreducible right Kepler half-walls with n bricks is C_{n-1} by giving a bijection between these walls and the set of Dyck paths from $(0, 0)$ to $(2n, 0)$ which touch the x-axis only at their endpoints. Although it follows immediately that there are C_n right Kepler half-walls with n bricks, we prove this result by extending our original bijection to right Kepler half-walls. In the same way, we give a bijection between Kepler walls with n bricks whose first row has one brick space and lattice paths from $(0,0)$ to $(2n,0)$ which use only up $(1,1)$ and down $(1,-1)$ steps and which begin with an up step. This shows that there are $\binom{2n-1}{n}$ such Kepler walls.

In Section 3 we use our bijection from Section 2 to study Kepler walls with bounded width. Throughout the paper we number the columns of a Kepler wall so that the column containing the unique top row brick is column 0, and the other columns are numbered in increasing order from left to right. We first show that the ordinary generating function for Kepler walls whose bricks lie weakly between columns 0 and m is

$$\frac{U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)},$$

for all $m \geq 0$. Here $U_n(x)$ is the nth Chebyshev polynomial of the second kind, which may be defined by $U_n(\cos t) = \dfrac{\sin((n+1)t)}{\sin(t)}$. We then show that for any $k, m \geq 0$, the ordinary generating function for Kepler walls whose bricks lie weakly between columns $-k$ and m is

$$\frac{U_{k+1}\left(\frac{1}{2\sqrt{x}}\right)U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)}{U_{m+k+2}\left(\frac{1}{2\sqrt{x}}\right)}.$$

Using these results we show that the number of Kepler walls with n bricks in which every brick is in columns 0, 1, or 2 is F_{2n-2}, the number in which every brick is in columns -1, 0, or 1 is F_{2n-1}, and the number in which every brick is in columns -1, 0, 1, or 2 is 3^{n-1}. Here F_n is the nth Fibonacci number, with initial conditions $F_0 = F_1 = 1$. In the same way, we show that the number of Kepler walls with n bricks in which every brick is in columns -2, -1, 0, 1, 2, or 3 is equal to the number of order-consecutive partitions of n (see [1]).

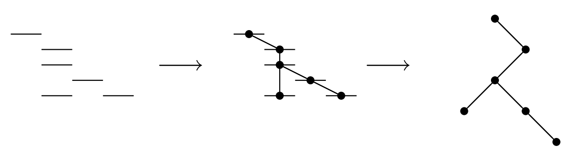
In Section 4 we study Kepler walls with n bricks which have k distinguishable top row brick spaces. We first use our bijection from Section 2 to construct a bijection between these Kepler walls and the set of lattice paths from $(0,0)$ to $(n+k-2,n)$, thus showing that there are $\binom{2n+k-2}{n}$ of these walls. Next we consider Kepler walls with k distinguishable top row brick spaces, some of which are required to contain bricks. We give a bijection which reduces this case to the previous one, thus showing that these walls are also counted by a binomial coefficient. We conclude the section by using inclusion-exclusion to enumerate Kepler walls whose top row consists of two bricks separated by a given number of empty brick spaces.

In the last section we study Kepler walls with k distinguishable top row brick spaces which lie between two given columns. Viewing such walls as sequences of rows of bricks, we explain how to use the transfer matrix method to enumerate these walls based on their top and bottom rows. We use this technique to show that the number of Kepler walls with n bricks and at most 3 columns is F_{2n+1}, while the number of Kepler walls with n bricks and at most 4 columns is $\frac{3^{n+1}-1}{2}$. We also use this technique to show that the number of Kepler walls with n bricks and at most 5 columns is the number of walks of length $2n+5$ in the path P_6 which begin at the leftmost vertex and end at the rightmost vertex.

2 Walls Below One Brick

A (nonempty) *binary tree* is a tree with a unique root node, in which every node may have a left child, a right child, both, or neither. A *Dyck path* is a lattice path which begins at $(0,0)$, ends on the x-axis, uses only up $(1,1)$ and down $(1,-1)$ steps, and never passes below the x-axis. We sometimes find it convenient to view a Dyck path as a sequence of Us and Ds, corresponding to up and down steps, respectively. We say a Dyck path is *irreducible* whenever it only touches the x-axis at its starting and ending points. It is well known that the number of binary trees with n nodes and the number of Dyck paths which end at $(2n,0)$ are both the Catalan number C_n. In this section we use binary trees and Dyck paths in the process of enumerating Kepler walls with a single brick in their top rows. We begin by giving a bijection between irreducible right half-walls and binary trees whose roots have no left subtree.

Suppose we have an irreducible right half-wall W on n bricks. The bricks in W will become the nodes in W's binary tree, T. The parent of a brick b is the brick the fewest rows above b in either b's column or the column immediately to its left. In T, a node p corresponding to a brick p' will have a child v whenever v's brick, v', is p''s child; v will be a left child if v' is in p''s column, and a right child if v' is in the column to the right of p'. It is easy to see that T is a binary tree whose root node does not have a left child. The example below illustrates this process.



To find the irreducible Dyck path which corresponds to T, we use a recursively defined map α from binary trees on n nodes to Dyck paths to $(2n,0)$. When $n=0$, we must have $\alpha(\emptyset)=\emptyset$, and when $n=1$ we must have $\alpha(\cdot)=UD$. Now assume we have defined α from binary trees with k nodes to Dyck paths with $2k$ steps for all $k<n$. Let T denote a binary tree with $n\geq 2$ nodes, let T_r denote the right subtree of the root of T and let T_l denote the left subtree of the root of T. Then we define $\alpha(T)=U\alpha(T_r)D\alpha(T_l)$. It is routine to prove that α is a bijection, and we observe that $\alpha(T)$ is an irreducible Dyck path if and only if the left subtree of the root of T is empty, which occurs if and only if W is an irreducible right half-wall.

Definition 2.1 *For any irreducible right Kepler half-wall W with n bricks, we write $\varphi(W)$ to denote the irreducible Dyck path with $2n$ steps obtained from W by the above process.*

Theorem 2.2 *For all $n\geq 0$, the map φ is a bijection between irreducible right Kepler half-walls with n bricks and irreducible Dyck paths with $2n$ steps.*

Proof. Let γ denote the map described above from the set of irreducible right half-walls to the set of binary trees for which $\varphi=\alpha\circ\gamma$. Since α is a bijection, it is sufficient to show that

γ is also a bijection. To do this, for each binary tree whose root has no left subtree we define an irreducible right half-wall $\gamma^{-1}(T)$; we then show that $\gamma^{-1}(\gamma(W)) = W$ for any irreducible right half-wall W and $\gamma(\gamma^{-1}(T)) = T$ for any binary tree T whose root has no left subtree.

Suppose T is a binary tree whose root has no left subtree. The root node of T will correspond to the root brick of $\gamma^{-1}(T)$. To construct the rest of $\gamma^{-1}(T)$, perform a depth-first search of T with right priority, inserting a new brick in $\gamma^{-1}(T)$ each time you search a node of T. If a node p in T has a left (resp. right) child v, place v's brick in the lowest supported position in p's brick's column (resp. the column immediately to the right of p's brick's column).

To show that $\gamma(\gamma^{-1}(T)) = T$ for any binary tree T whose root has no left subtree, suppose T is such a binary tree. Since T and $\gamma(\gamma^{-1}(T))$ have the same number of nodes, we identify nodes in $\gamma(\gamma^{-1}(T))$ with nodes in T via $\gamma \circ \gamma^{-1}$. Now suppose b is a right child of p in T. When applying γ^{-1}, we place b's brick immediately after placing p's brick, so b will be a right child of p in $\gamma(\gamma^{-1}(T))$. Similarly, suppose b is a left child of p in T. Every brick we place between placing p's brick and placing b's brick will be placed in a column strictly to the right of p's brick's column. Therefore b will be a left child of p in $\gamma(\gamma^{-1}(T))$. Since T and $\gamma(\gamma^{-1}(T))$ have the same nodes and same relations between nodes, $\gamma(\gamma^{-1}(T)) = T$.

To show that $\gamma^{-1}(\gamma(W)) = W$ for any irreducible right half-wall, suppose W is such a wall with $\gamma^{-1}(\gamma(W)) \neq W$. Note that W and $\gamma^{-1}(\gamma(W))$ have the same number of bricks, and identify bricks in W with bricks in $\gamma^{-1}(\gamma(W))$ via $\gamma^{-1} \circ \gamma$. Similarly, identify nodes in $T = \gamma(W)$ with bricks in W via γ. Consider the first node v in our depth-first search of T which occurs in different positions in $\gamma^{-1}(\gamma(W))$ and W.

If v is a right child of a node p in T then in both W and $\gamma^{-1}(\gamma(W))$, the brick v lies below p and in the column immediately to the right of p's column. In $\gamma^{-1}(\gamma(W))$, the brick v is in the lowest supported position in its column, so in W the brick v must be in a higher position in the same column. Since $\gamma^{-1}(\gamma(W))$ is a Kepler wall, v is supported by some brick in $\gamma^{-1}(\gamma(W))$. If v is supported by a brick in the column to its left, then v is supported by its parent in T. In this case v is above its parent in W, which is a contradiction. If v is supported by a brick in its own column then v is a left child of that brick in T, which is a contradiction. If v is supported by a brick u in the column to its right, then in W the brick v lies above u. In this case the highest brick in u's column which is below v in W is a child of v in T. But this brick was inserted in $\gamma^{-1}(\gamma(W))$ before v, which is a contradiction.

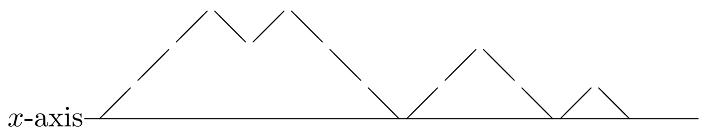
A similar argument shows that v cannot be a left child o a node in T, and it follows that $\gamma^{-1}(\gamma(W)) = W$. □

Notice that we now know that there are also C_n Kepler half-walls with n bricks: we can form a bijection between irreducible half-walls with n bricks and half-walls with $n-1$ bricks by simply removing the single brick in the leftmost column of the irreducible half-wall. However, we obtain a different proof of this fact by extending the map φ to a map from right Kepler half-walls to Dyck paths. To describe this extension, note that we can decompose a given right Kepler half-wall W into a sequence of irreducible half-walls $W_1, W_2, \ldots W_k$ by applying the parenting algorithm to all bricks except those in the leftmost column. We see that the ith brick in the left column is the root of some irreducible right Kepler half-wall, W_i. To construct the Dyck path P from $(0,0)$ to $(2n,0)$, apply φ to each W_i and place the resulting paths consecutively along the x-axis.

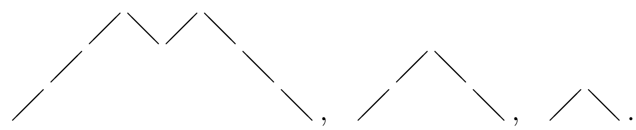
Definition 2.3 *For any right Kepler half-wall W with n bricks, we write $\phi(W)$ to denote the Dyck path with $2n$ steps obtained from W by the process described above.*

Theorem 2.4 *The map ϕ is a bijection from right Kepler half-walls with n bricks to Dyck paths with $2n$ steps.*

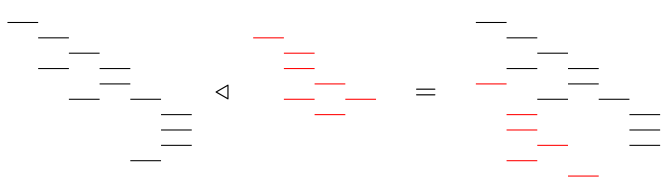
Proof. First note that every Dyck path P can be uniquely decomposed into a sequence of irreducible pieces. For example, the path



can be split into three irreducible Dyck paths:



Decompose P into its irreducible pieces $P_1, \ldots, P_k$, and find $\varphi^{-1}(P_i)$ for all i, $1 \le i \le k$. To construct $\phi^{-1}(P)$, we will begin with $\varphi^{-1}(P_1)$ and glue on $\varphi^{-1}(P_2), \varphi^{-1}(P_3), \ldots, \varphi^{-1}(P_k)$ successively. Place the root brick of $\varphi^{-1}(P_2)$ in the lowest supported position in the 0th column of $\varphi^{-1}(P_1)$, and place the top brick in $\varphi^{-1}(P_2)$'s first column in the lowest supported position in the first column of $\varphi^{-1}(P_1)$. Now, place all bricks in the first column whose support in $\varphi^{-1}(P_2)$ comes only from the first or 0th columns. In general, place the top brick in the jth column of $\varphi^{-1}(P_2)$ in the lowest supported position in the jth column of $\varphi^{-1}(P_1)$, and place all bricks from the columns less than or equal to j and whose supporting bricks from $\varphi^{-1}(P_2)$ are already in place. Append $\varphi^{-1}(P_3), \ldots, \varphi^{-1}(P_k)$ in a similar fashion. We give an example of this gluing process below.



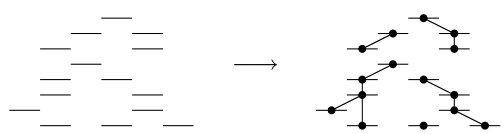
Note that if we apply the parenting algorithm to the bricks in $\phi^{-1}(P)$ belonging to $\varphi^{-1}(P_i)$ and to $\varphi^{-1}(P_i)$ itself, we obtain the same parental relationships. So if we apply ϕ to $\phi^{-1}(P)$, we will recover the sequence of irreducible half-walls $\varphi^{-1}(P_1), \ldots, \varphi^{-1}(P_k)$. Since φ is a bijection, we have $\phi(\phi^{-1}(P)) = P$. Similarly, applying ϕ to a wall W will first give us a sequence of irreducible walls, $W_1, \ldots, W_k$. If we then apply ϕ^{-1} to $\phi(W)$, we will recover the same sequence of irreducible half-walls upon applying φ^{-1} to the irreducible pieces of $\phi(W)$. Thus, $\phi^{-1}(\phi(W)) = W$. $\square$

Corollary 2.5 *The number of Kepler half-walls with n bricks is the nth Catalan number C_n.*

Proof. This is immediate from Theorem 2.2. □

We are now ready to look at a more general case: Kepler walls with one brick in their top row. We will define a function from the set of Kepler walls below one brick to the set of lattice paths from $(0, 0)$ to $(2n, 0)$ that use only the steps $(1, 1)$ and $(1, -1)$ and begin with an up-step. First, note that we can decompose such a path into irreducible pieces above and below the x-axis, just as we decomposed a Dyck path into its irreducible pieces.

This decomposition lends itself to the following map. Given a Kepler wall W, apply the parenting algorithm described before Definition 2.1 to all bricks in columns with index $i > 0$. For the bricks in the columns with index less than -1, apply the following modification of the parenting algorithm: for a brick b in column i, look above b in columns i and $i + 1$ and assign the brick p that is closest in height to b to be b's parent. We have now partitioned our wall W into a collection of irreducible right half-walls and irreducible left half-walls. We give an example below.



Call the irreducible half-wall whose root brick lies in the first row W_1. Find the next lowest brick in the 0th or -1st column and call its corresponding irreducible half-wall W_2. Continue in this manner to get a sequence $W_1, W_2, \ldots, W_k$ of irreducible half-walls. Apply φ to each W_i, first reflecting W_i over a vertical line whenever W_i is a left half-wall. We now have a sequence of irreducible paths $\varphi(W_1), \ldots, \varphi(W_k)$. Line these paths up consecutively beginning at $(0, 0)$, reflecting $\varphi(W_i)$ over the x-axis whenever W_i is a left half-wall. By construction, we have a lattice path with $2n$ steps.

Definition 2.6 *For any Kepler wall W with n bricks, we write $\Phi(W)$ to denote the lattice path with $2n$ steps obtained from W by the process described above.*

Theorem 2.7 *The map Φ is a bijection from the set of Kepler walls with n bricks to the set of lattice paths from $(0, 0)$ to $(2n, 0)$ which begin with an up step.*

Proof. Let P be a path of the given type. To build $\Phi^{-1}(P)$, we first split P into its sequence of irreducible Dyck paths $P_1, \ldots, P_k$ that lie above and below the x-axis. We apply φ^{-1} to each P_i, reflecting each $\varphi^{-1}(P_i)$ vertically whenever P_i lies below the x-axis. We paste these half-walls together as in the proof of Theorem 2.4: let the root brick of $\varphi^{-1}(P_1)$ be the root of $\Phi^{-1}(P)$. In general, if P_i lies above the x-axis, place the root of $\varphi^{-1}(P_i)$ in the lowest supported position in the 0th column of $\Phi^{-1}(P)$ and use the gluing algorithm exactly as in the proof of Theorem 2.4. If P_i lies below the x-axis, place the root of $\varphi^{-1}(P_i)$ in the lowest supported position in the -1st column of $\Phi^{-1}(P)$. For a general $j \geq 0$, place the top brick in the jth column of $\varphi^{-1}(P_i)$ in the lowest supported position in the $-j - 1$st column of Φ^{-1} and place the bricks that lie in a column with index less than or equal to j and whose support is already in place. This process is the "reflected" gluing algorithm and is directly analogous to the original gluing algorithm used in the proof of Theorem 2.4.

By construction, if we apply the modified parenting algorithm to $\Phi^{-1}(\Phi(W))$, we will see the same parental relationships we saw in W, just as we did in the proof of Theorem 2.4. Thus, $\Phi^{-1}(\Phi(W)) = W$ and a similar argument shows that $\Phi(\Phi^{-1}(P)) = P$ for any path P. □

Corollary 2.8 *The number of Kepler walls with n bricks which have one brick in the top row is $\binom{2n-1}{n}$.*

Proof. This is immediate from Theorem 2.7. □

3 Walls of Bounded Width

To this point we have allowed our Kepler walls to have any width. In this section we study Kepler walls of restricted width, which are required to lie between fixed columns. To start, we show that for any Kepler wall W, the map Φ translates column numbers of bricks in W to heights of steps in $\Phi(W)$.

Proposition 3.1 *Suppose m and k are nonnegative integers and W is a Kepler wall. Then the bricks of W lie weakly between columns $-k$ and m if and only if $\Phi(W)$ does not cross the lines $y = -k$ and $y = m + 1$.*

Proof. Assign numbers to the nodes of a binary tree recursively so that the root has number 1, and if a node has number a then its left child has number a and its right child has number $a + 1$. It is routine to show that if a right Kepler half-wall W has binary tree T then the numbers on the nodes of T are each one more than the column numbers of the associated bricks in W. It is also routine to show by induction that if T is a binary tree then the largest number on any node of T is the maximum height of the associated Dyck path $\alpha(T)$. Now the result follows from the construction of Φ. □

Proposition 3.1 allows us to use the combinatorics of lattice paths to obtain generating functions for Kepler walls which lie between a given pair of columns. Our expressions for these generating functions involve Chebyshev polynomials of the second kind, which are defined by $U_n(\cos t) = \frac{\sin((n+1)t)}{\sin t}$. It is routine to show that $U_{-1}(x) = 0$, $U_0(x) = 1$, and

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \qquad (n \geq 1). \tag{1}$$

We begin by considering Kepler half-walls of bounded width.

Theorem 3.2 *For all $m \geq 0$, let $F_m(x)$ denote the ordinary generating function for the Kepler walls in which every brick is weakly between columns 0 and m. Then*

$$F_m(x) = \frac{U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_{m+2}\left(\frac{1}{2\sqrt{x}}\right)}. \tag{2}$$

Proof. We argue by induction on m. For each $n \geq 0$ there is exactly one Kepler wall with n bricks in which every brick is in column 0, so

$$F_0(x) = \frac{1}{1-x} = \frac{U_1\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_2\left(\frac{1}{2\sqrt{x}}\right)}.$$

Therefore (2) holds for $m = 0$.

Now suppose $m \geq 1$. In view of Proposition 3.1, $F_m(x)$ is the generating function for Dyck paths which do not cross the line $y = m + 1$. For each such (nonempty) path π there are unique paths π_1 and π_2 such that $\pi = U\pi_1 D\pi_2$, the path π_1 does not cross $y = m$, and π_2 does not cross $y = m + 1$. Therefore $F_m(x) = 1 + xF_{m-1}(x)F_m(x)$, and by induction we have

$$\begin{aligned} F_m(x) &= \frac{1}{1 - xF_{m-1}(x)} \\ &= \frac{U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)}{U_{m+1}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}U_m\left(\frac{1}{2\sqrt{x}}\right)} \\ &= \frac{U_{m+1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_{m+2}\left(\frac{1}{2\sqrt{x}}\right)}, \end{aligned}$$

where the last step follows from (1). □

Corollary 3.3 *For all $n \geq 1$, the number of Kepler walls with n bricks in which every brick is in columns 0, 1, or 2 is F_{2n-2}. Here F_n is the nth Fibonacci number, with initial conditions $F_0 = F_1 = 1$.*

Proof. Set $m = 2$ in (2) to obtain

$$F_2(x) = \frac{1 - 2x}{1 - 3x + x^2},$$

which is the generating function for $1, F_0, F_2, F_4, \ldots$. □

We now turn our attention to arbitrary width-restricted Kepler walls. To simplify our generating function in this case we need the following well known identity involving Chebyshev polynomials.

Lemma 3.4 *For all $k, l \geq -1$ we have*

$$U_{k+1}U_{l+1} - U_kU_l = U_{k+l+2}. \tag{3}$$

Here we abbreviate $U_* = U_*\left(\frac{1}{2\sqrt{x}}\right)$.

Proof. We argue by induction on k. Line (3) is immediate when $k = -1$ and follows from (1) when $k = 0$, so suppose $k \geq 1$. Use (1) to eliminate U_{k+1} and U_k on the left side of (3). Now use induction and (1) twice each to obtain (3). □

Theorem 3.5 *For all $k, m \geq 0$, let $F_{k,m}(x)$ denote the ordinary generating function for the Kepler walls in which every brick is weakly between columns k and m. Then*

$$F_{k,m}(x) = \frac{U_{k+1}U_{m+1}}{U_{m+k+2}}, \tag{4}$$

where we have abbreviated $U_* = U_*\left(\frac{1}{2\sqrt{x}}\right)$.

Proof. In view of Proposition 3.1, $F_{k,m}(x)$ is the generating function for lattice paths which begin with an up step and which do not cross the lines $y = -k$ and $y = m + 1$. Each such (nonempty) path consists of an alternating sequence of positive and negative Dyck paths in which only the last path may be empty. Therefore

$$\begin{aligned} F_{k,m}(x) &= 1 + F_{k-1}(x) \sum_{j=0}^{\infty} (F_m(x) - 1)^{j+1} (F_{k-1}(x) - 1)^j \\ &= 1 + \frac{(F_m(x) - 1) F_{k-1}(x)}{1 - (F_m(x) - 1)(F_{k-1}(x) - 1)} \\ &= \frac{\sqrt{x} U_{m+1} U_{k+1}}{\sqrt{x}(U_{m+2} U_k + U_{m+1} U_{k+1}) - U_{m+1} U_k}, \end{aligned}$$

where the last step follows from (2). To simplify the denominator, use (1) to evaluate $\sqrt{x} U_{m+1} U_{k+1} - U_{m+1} U_k$ and then use (3) to simplify the result. □

We conclude this section by using Theorem 3.5 to enumerate several families of width restricted Kepler walls.

Corollary 3.6 *For all $n \geq 1$, the number of Kepler walls with n bricks in which every brick is in columns -1, 0, or 1 is F_{2n-1}. Here F_n is the nth Fibonacci number, with initial conditions $F_0 = F_1 = 1$.*

Proof. Set $m = k = 1$ in (4) to obtain

$$F_{1,1}(x) = \frac{(1-x)^2}{1 - 3x + x^2},$$

which is the generating function for $1, F_1, F_3, F_5, \ldots$. □

Corollary 3.7 *For all $n \geq 1$, the number of Kepler walls with n bricks in which every brick is in columns -1, 0, 1, or 2 is 3^{n-1}.*

Proof. Set $m = 2$ and $k = 1$ in (4) to obtain

$$F_{2,1}(x) = \frac{1 - 2x}{1 - 3x},$$

which is the generating function for $1, 1, 3, 9, 27, \ldots$. □

Corollary 3.8 *For all $n \geq 1$, the number of Kepler walls with n bricks in which every brick is in columns -2, -1, 0, 1, 2, or 3 is equal to the number of order-consecutive partitions of n (see [1]). This is sequence A007052 in [3].*

Proof. Set $m = 3$ and $k = 2$ in (4) to obtain

$$F_{3,2}(x) = \frac{1 - 3x + x^2}{1 - 4x + 2x^2},$$

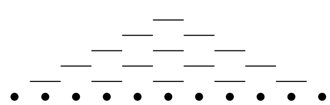
which is the generating function for the number of order-consecutive partitions of n. □

4 Walls Below Many Bricks

In Sections 2 and 3 we studied Kepler walls with a single top row brick; in this section we turn our attention to Kepler walls whose top rows may have several bricks. In other words, we now consider Kepler walls whose top rows have several brick spaces. We will find it convenient throughout to consider these spaces to be distinguishable. This means, for instance, that if the top row has five brick spaces, then it supports $5 = \binom{5}{1}$ Kepler walls with exactly one brick, $21 = \binom{7}{3}$ walls with exactly two bricks, and $126 = \binom{9}{5}$ walls with exactly three bricks.

As these examples suggest, Kepler walls whose top rows have a given number of brick spaces are counted by certain binomial coefficients. To prove this, we use our previous bijections to give a constructive bijection between these new Kepler walls and an appropriate set of lattice paths.

To begin, suppose $k \geq 3$ is odd, set $k = 2m + 1$, and suppose W is a wall whose top row has k brick spaces. To construct the associated lattice path, first stack a pyramid of $m + (m-1) + \ldots + 2 + 1 = \binom{m+1}{2}$ bricks atop the first row of W by placing bricks above spaces $2, 4, 6, \ldots, 2m$, and in the row above that bricks above spaces $3, 5, \ldots, 2m-1$, etc. The figure below shows the result of doing this when $m = 5$; here the dots in the bottom row represent the brick spaces in the top row of W.



The resulting wall W' is an ordinary Kepler wall with $n + \binom{m+1}{2}$ bricks, so $\Phi(W')$ is a lattice path with $n + \binom{m+1}{2}$ up steps and $n + \binom{m+1}{2}$ down steps. Moreover, by the construction of W' and the definition of Φ the lattice path $\Phi(W')$ has the form

$$U^m \pi_1 D^m D^{m-1} \pi_2^c U^{m-1} U^{m-2} \pi_3 D^{m-2} \cdots D^2 \pi_{m-1}^c U^2 U \tau$$

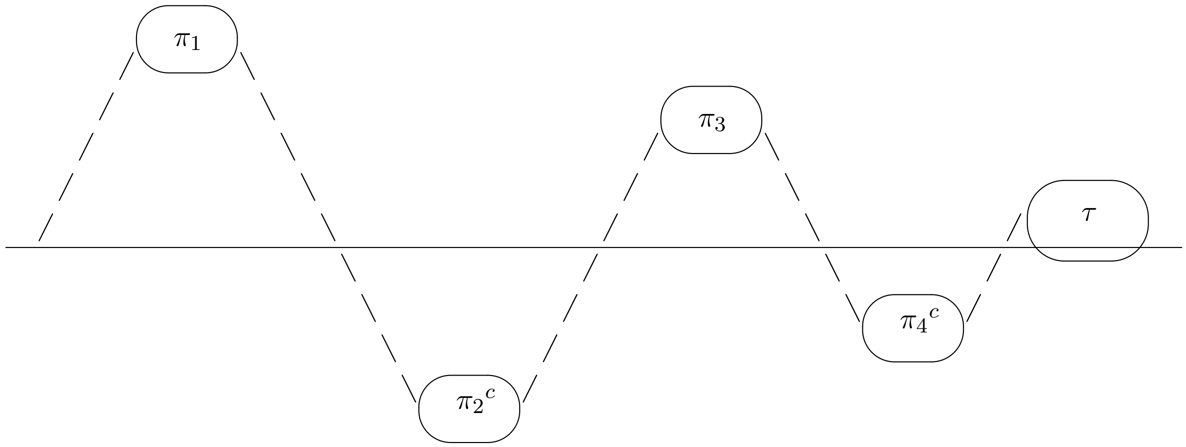
if m is odd and

$$U^m \pi_1 D^m D^{m-1} \pi_2^c U^{m-1} U^{m-2} \pi_3 D^{m-2} \cdots U^2 \pi_{m-1} D^2 D \tau^c$$

if m is even, where

- $\pi_1, \ldots, \pi_{m-1}$ are lattice paths which begin and end at height 0 and do not pass below height -1,
- τ is a lattice path which begins at height 0 and ends at height -1, and
- π^c is the reflection of the lattice path π over the x-axis.

The diagram below shows $\Phi(W')$ when $m = 5$.



To obtain an appropriate lattice path from $\Phi(W')$, we replace each string of the form $U^{m+1-i}\pi_i D^{m+1-i}$ with the string $\pi_i D^2$ and we replace each string of the form $D^{m+1-i}\pi_i{}^c U^{m+1-i}$ with the string $\pi_i D^2$. Geometrically, this amounts to reflecting each piece of the path that lies below the x-axis over the x-axis and removing certain steps that are forced by the pyramid construction. Note that the resulting lattice path has the form $\pi_1 D^2 \pi_2 D^2 \cdots \pi_{m-1} D^2 \tau$. This process removes $m+(m-1)+\ldots+2+1 = \binom{m+1}{2}$ up steps and $(m-2)+(m-3)+\ldots+2+1 = \binom{m-1}{2}$ down steps, so the resulting lattice path has exactly n up steps and $n+k-2$ down steps.

Definition 4.1 *Suppose $k \geq 1$ is odd and $n \geq 0$. For each Kepler wall W with k brick spaces in its top row and n bricks, we write $\Psi(W)$ to denote the lattice path with n up steps and $n+k-2$ down steps obtained by the process described above.*

Theorem 4.2 *Suppose $k \geq 1$ is odd and $n \geq 0$. The map Ψ is a bijection from the set of Kepler walls with k top row brick spaces and n bricks to the set of lattice paths with n up steps and $n+k-2$ down steps.*

Proof. To show Ψ is a bijection, we construct Ψ^{-1}. To this end, suppose π is a lattice path with n up steps and $n+k-2$ down steps, and fix m so that $k = 2m+1$. We claim there exist unique lattice paths $\pi_1, \ldots, \pi_{m-1}, \tau$ such that

(i) $\pi_1, \ldots, \pi_{m-1}$ begin and end at height 0 and never pass below height -1,

(ii) τ begins at height 0 and ends at height -1, and

(iii) $\pi = \pi_1 D^2 \pi_2 D^2 \cdots \pi_{m-1} D^2 \tau$.

To prove this claim, first note that if $k = 3$ then $m = 1$ so in this case we must take $\tau = \pi$. Now observe that if $k > 3$ then π must pass below height -1, and the first such crossing must consist of two consecutive down steps. Let π_1 denote the part of π preceding the first of these down steps, so that $\pi = \pi_1 D^2 \pi'$. Note that by construction π_1 is a lattice path which begins and ends at height 0 and does not pass below height -1 and π' is a lattice path with $k-4$ more down steps than up steps. Therefore, by induction on k there exist lattice paths $\pi_2, \ldots, \pi_{m-1}, \tau$ such that conditions (i)–(iii) above hold, and our claim follows.

Now set

$$\eta = U^m \pi_1 D^m D^{m-1} {\pi_2}^c U^{m-1} U^{m-2} \pi_3 D^{m-2} \cdots D^2 \pi_{m-1}^c U^2 U \tau$$

if m is odd and

$$\eta = U^m \pi_1 D^m D^{m-1} {\pi_2}^c U^{m-1} U^{m-2} \pi_3 D^{m-2} \cdots U^2 \pi_{m-1} D^2 D \tau^c$$

if m is even. By construction $\Phi^{-1}(\eta)$ is a Kepler wall whose first m rows are full; we define $\Psi^{-1}(\pi)$ to be the Kepler wall obtained by removing these rows.

It is routine to verify that $\Psi^{-1}(\Psi(W)) = W$ for any Kepler wall W with k top row brick spaces and $\Psi(\Psi^{-1}(\pi)) = \pi$ for any lattice path π with n up steps and $n+k-2$ down steps. □

When $k = 2m$ is even and W is a Kepler wall with k top row brick spaces, we may again construct a lattice path $\Psi(W)$ with n up steps and $n+k-2$ down steps. This construction is analogous to the construction of $\Psi(W)$ when k is odd, except in this case $\Psi(W')$ begins with $U^m D \pi_1 D^m$ instead of $U^m \pi_1 D^m$ and π_1 begins at height -1 and ends at height 0. Thus we can enumerate Kepler walls with k top row brick spaces for any k.

Corollary 4.3 *For all $n \geq 0$ and all $k \geq 1$ there are exactly $\binom{2n+k-2}{n}$ Kepler walls with k top row spaces and n bricks.*

Proof. This is immediate from Theorem 4.2 and the discussion following its proof. □

Having enumerated Kepler walls with k top row brick spaces, any or all of which may be empty, we now turn our attention to Kepler walls in which some of these top row brick spaces are required to contain bricks. Notice that if a brick space adjacent to an end brick space in a row of brick spaces is required to contain a brick then the corresponding end brick space must be empty. With this in mind, we say a set A of brick spaces in a row is *feasible* whenever no two brick spaces in A are adjacent and A does not contain either of the brick spaces adjacent to the end spaces in the row. Next we show that for any feasible set of top row brick spaces, Kepler walls in which each brick space in the set contains a brick are also enumerated by binomial coefficients.

Theorem 4.4 *Fix $m, k \geq 0$, let A denote a feasible set of m brick spaces in a row of $m+k$ brick spaces, and let d denote the number of brick spaces in A at the ends of the row. Then there is a constructive bijection between the set of Kepler walls with $m+k$ top row brick spaces and $n+m$ bricks in which every space in A contains a brick and the set of Kepler walls with $m+k+d$ top row brick spaces and n bricks.*

Proof. Suppose W is a Kepler wall with $m+k$ top row brick spaces and $n+m$ bricks in which every space in A contains a brick. Color each brick in W red or blue, beginning with the bricks in the top row and moving down, as follows.

- Color each brick which occupies a space in A red and color all other bricks in the top row blue.

- If a brick is supported by at least one blue brick then color it blue; color all other bricks red.

To obtain a new wall from W, first remove all of the red bricks from the top row, and then move each remaining red brick up one row. Note that every brick not in the top row is still supported by a brick in a row above. In addition, this process cannot create pairs of adjacent bricks, since moving a red brick to a brick space adjacent to a blue brick can only occur if the red brick were originally supported by the now adjacent blue brick. But such a brick would have been colored blue and not been moved. It follows that the resulting object is a Kepler wall with $m+k+d$ top row brick spaces and n bricks.

To reverse this process, suppose W is a Kepler wall with $m+k+d$ top row brick spaces and n bricks. Place a row of $m+k$ empty brick spaces above the top row of W and insert m bricks in the spaces in A. Color these bricks red. Color each remaining brick red or blue, moving down one row at a time, so that every brick which is supported by a red brick is colored red and all remaining bricks are colored blue. To obtain a new wall from W, move each blue brick up one space. As before, the resulting object is a Kepler wall with $m+k$ top row brick spaces and $n+m$ bricks in which every space in A contains a brick.

It is routine to verify that these two processes are inverses of one another. □

Theorem 4.4 and Corollary 4.3 allow us to enumerate Kepler walls restricted by any feasible set.

Corollary 4.5 *Let A denote a feasible set of m brick spaces in a row of $m+k$ brick spaces, and let d denote the number of brick spaces in A at the ends of the row. Then there are exactly $\binom{2n+m+k+d-2}{n}$ Kepler walls with $m+k$ top row brick spaces and $n+m$ bricks in which every space in A contains a brick.*

Proof. This is immediate from Theorem 4.4 and Corollary 4.3. □

Corollary 4.6 *Fix $m \geq 1$. For all $n \geq 0$ there are exactly $\binom{2n-1}{n-m}$ Kepler walls with n bricks whose top row contains m bricks and $m-1$ empty brick spaces.*

Proof. Set $k = m-1$ and $d = 2$ and replace n with $n-m$ in Corollary 4.5. □

Corollary 4.7 *Fix $k \geq 1$. For all $n \geq 0$ there are exactly $\binom{2n+k-2}{n-2}$ Kepler walls with n bricks whose top row has a brick at each end and k additional brick spaces.*

Proof. Set $m = d = 2$ and replace n with $n-2$ in Corollary 4.5. □

Corollary 4.5 also allows us to use inclusion-exclusion to enumerate Kepler walls with a predetermined top row. As an example, we enumerate Kepler walls whose top row consists of two bricks with k empty brick spaces between them.

Theorem 4.8 *For all $k \geq 1$ and all $n \geq 0$ there are*

$$\sum_{r=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^r \binom{k-r-1}{r} \binom{2n-2r+k-2}{n+k-r}$$

Kepler walls with exactly n bricks whose top row consists of two bricks with k empty brick spaces between them.

Proof. Fix r, $0 \leq r \leq \lfloor \frac{k-1}{2} \rfloor$, and consider a row of $k+2$ brick spaces. Let A denote a feasible set of $r+2$ brick spaces in this row which contains both end spaces. Set $d = 2$ and $m = r+2$ and replace n with $n-r-2$ and k with $k-r$ in Corollary 4.5 to see that there are exactly $\binom{2n-2r+k-2}{n+k-r}$ Kepler walls with bricks in all of the brick spaces in A. Since there are $\binom{k-r-1}{r}$ choices for A, by inclusion-exclusion there are

$$\sum_{r=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^r \binom{k-r-1}{r} \binom{2n-2r+k-2}{n+k-r}$$

Kepler walls with n bricks in which none of the brick spaces between the two bricks in the top row contains a brick, as desired. □

5 Walls of Bounded Width Below Many Bricks

In Section 3 we studied Kepler walls whose top row contains a single brick, and all of whose bricks lie between two given columns. In Section 4 we studied Kepler walls whose top row contains several brick spaces, but which have no width restriction. In this section we use the transfer matrix method to enumerate Kepler walls whose bricks lie between two given columns, and whose top row may consist of any number of bricks which will fit legally between these two columns. To set the stage, we give a weighted directed graph structure to the set of possible rows in a Kepler wall of width w.

Definition 5.1 *For any $w \geq 1$, we write V_w to denote the set of nonempty rows of w bricks in which no two bricks are adjacent. For all $i, j \in V_w$, we write $i \to j$ to indicate that when we place i above j, every brick in j is supported by some brick in i. For all $i \in V_w$, we write* $\mathrm{br}(i)$ *to denote the number of bricks in i. The* Kepler graph of width w *is the edge-weighted directed graph with vertex set V_w whose adjacency matrix A_w is given by*

$$(A_w)_{ij} = \begin{cases} x^{\mathrm{br}(j)-1} & \text{if } i \to j; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose we are building a Kepler wall of width w by inserting new rows at the bottom of a partial wall. Then the Kepler graph of width w describes how we may construct our wall, in the sense that if the bottom row of our current wall is row i, then we may insert a row j below i if and only if $i \to j$. This observation allows us to formulate the following expression for the generating function for Kepler walls of width at most w.

Proposition 5.2 *Suppose $w \geq 1$ and $i, j \in V_w$. Then the generating function for Kepler walls of width at most w which begin with row i and end with row j is*

$$\sum_{n=1}^{\infty} x^{\mathrm{br}(i)} (A_w^{n-1})_{ij} x^{n-1}. \tag{5}$$

Proof. Each such wall corresponds to a path in the Kepler graph of width w which begins at i and ends at j: for each vertex in our path, we continue construction of our wall by making that vertex the new bottom row. Each edge in such a path has a weight whose exponent is one less than the number of bricks the new row contributes, so the total number of bricks in the wall is the sum of the number of edges in the path, the number of bricks in i, and the exponents in the weights on the edges in the path. Thus the total weight of all walls with n rows is the total weight of all paths with n vertices, which is $x^{\mathrm{br}(i)}(A_w^{n-1})_{ij}x^{n-1}$, as desired. □

To evaluate the generating function in Proposition 5.2, we recall the transfer matrix method, which we paraphrase from [4].

Proposition 5.3 *[4, Theorem 4.7.2] Suppose D is an edge-weighted directed graph with vertices $1, 2, \ldots, n$ and adjacency matrix A, so that A_{ij} is the weight of the edge from i to j. Then for all i, j with $1 \le i, j \le n$ we have*

$$\sum_{n=0}^{\infty}(A^n)_{ij}t^n = \frac{(-1)^{i+j}\det(I - tA; j, i)}{\det(I - tA)}, \qquad (6)$$

where $\det(B; j, i)$ is the determinant of the matrix obtained by removing the jth row and ith column of B and I is the $n \times n$ identity matrix.

In view of Propositions 5.2 and 5.3, the generating function for the number of Kepler walls with n bricks and at most w columns is rational for all $w \ge 1$. When w is small, these results allow us to find generating functions for a variety of collections of width-restricted Kepler walls.

Theorem 5.4 *For all $n \ge 0$, the number of Kepler walls with n bricks and at most 3 columns is F_{2n+1}. Here F_n is the nth Fibonacci number, with initial conditions $F_0 = F_1 = 1$.*

Proof. First note that if we order the elements of V_3 so that the second row in V_3 has a brick in its center space and the fourth row in V_3 has two bricks, then we have

$$A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & x \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & x \end{pmatrix}.$$

If we sum (5) over all i, j with $1 \le i, j \le 4$ and use (6) to simplify the result, we find that the ordinary generating function for the number of Kepler walls with n bricks and at most 3 columns is $\frac{x(1-x)}{1-3x+x^2}$. This is the generating function for $F_1, F_3, F_5, \ldots$. □

Theorem 5.5 *For all $n \ge 0$, the number of Kepler walls with n bricks and at most 4 columns is $\frac{3^{n+1}-1}{2}$.*

Proof. First note that if we order the elements of V_4 appropriately then we have

$$A_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & x & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & x & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & x & x & x \\ 1 & 1 & 1 & 1 & x & x & x \\ 1 & 1 & 1 & 1 & x & x & x \end{pmatrix}.$$

If we sum (5) over all i, j with $1 \le i, j \le 7$ and use (6) to simplify the result, we find that the ordinary generating function for the number of Kepler walls with n bricks and at most 4 columns is $\frac{1}{1-4x+3x^2}$. This is the generating function for $\left\{\frac{3^{n+1}-1}{2}\right\}_{n=0}^{\infty}$. □

Theorem 5.6 *For all $n \ge 0$, the number of Kepler walls with n bricks and at most 5 columns is the number of walks of length $2n+5$ in the path P_6 which begin at the leftmost vertex and end at the rightmost vertex. This is sequence A005021 in [3].*

Proof. This is similar to the proofs of Theorems 5.4 and 5.5, using the fact that for an appropriate ordering of V_5 we have

$$A_5 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & x & x & 0 & x & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & x & x & x & x & x & x & x^2 \\ 1 & 1 & 0 & 1 & 1 & 0 & x & x & x & x & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & x & x & x & x & x & x & x^2 \\ 1 & 1 & 1 & 1 & 1 & x & x & x & x & x & x & x^2 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & x & x & x & 0 \\ 1 & 1 & 1 & 1 & 1 & x & x & x & x & x & x & x^2 \end{pmatrix}. \qquad (7)$$

□

Theorem 5.7 *For all $n \ge 0$, the number of Kepler walls with n bricks, at most 5 columns, and in which the top row has a brick at each end and no other bricks, is equal to the number of walks of length $2n+1$ in the path P_6 with vertices $\{1, 2, 3, 4, 5, 6\}$ which begin at 1 and end at 4. This is sequence A094789 in [3].*

Proof. This is similar to the proofs of Theorems 5.4, 5.5, and 5.6, using (7). □

References

[1] F. K. Hwang and C. L. Mallows. Enumerating nested and consecutive partitions. *J. Combin. Theory, Ser. A*, 70(2):323–333, 1995.

[2] D. Knuth. Three Catalan bijections. Institut Mittag-Leffler preprint series, 2005, spring, #04; www.ml.kva.se/preprints/0405s.

[3] N. J. A. Sloane. The on-line encyclopedia of integer sequences. www.research.att.com/~njas/sequences.

[4] R. P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, 1997.