The Jacobi-Stirling Numbers

Eric S. Egge
(joint work with G. Andrews, L. Littlejohn, and W. Gawronski)

Carleton College

March 18, 2012
The Differential Operator $xD$

$$y = y(x)$$

$$D = \frac{d}{dx}$$
The Differential Operator $xD$

$y = y(x)$

$D = \frac{d}{dx}$

$xD[y] = 1xy'$

$(xD)^2[y] = 1xy' + 1x^2y^{(2)}$
The Differential Operator \( xD \)

\[
y = y(x) \quad \quad \quad D = \frac{d}{dx}
\]

\[
xD[y] = 1xy'
\]

\[
(xD)^2[y] = 1xy' + 1x^2y^{(2)}
\]

\[
(xD)^3[y] = 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)}
\]
The Differential Operator $xD$

\[ y = y(x) \]

\[ D = \frac{d}{dx} \]

\[ xD[y] = 1xy' \]

\[ (xD)^2[y] = 1xy' + 1x^2y^{(2)} \]

\[ (xD)^3[y] = 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)} \]

\[ (xD)^4[y] = 1xy' + 7x^2y^{(2)} + 6x^3y^{(3)} + 1x^4y^{(4)} \]
The Differential Operator $xD$

\[ y = y(x) \quad D = \frac{d}{dx} \]

\[ xD[y] = 1xy' \]
\[ (xD)^2[y] = 1xy' + 1x^2y^{(2)} \]
\[ (xD)^3[y] = 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)} \]
\[ (xD)^4[y] = 1xy' + 7x^2y^{(2)} + 6x^3y^{(3)} + 1x^4y^{(4)} \]

\[ (xD)^n[y] = \sum_{j=1}^{n} \left\{ \binom{n}{n+1-j} \right\} x^j y^{(j)} \]
\[ \left\{ \begin{array}{c} n \\ j \end{array} \right\} = \left\{ \begin{array}{c} n-1 \\ j-1 \end{array} \right\} + j \left\{ \begin{array}{c} n-1 \\ j \end{array} \right\} \]
The Jacobi-Stirling Numbers

\[ \ell_{\alpha,\beta}[y] = -(1 - x^2)y'' + (\alpha - \beta + (\alpha + \beta + 2)x)y' \]
\[ \ell_{\alpha,\beta}[y] = -(1 - x^2)y'' + (\alpha - \beta + (\alpha + \beta + 2)x)y' \]

**Theorem (Everitt, Kwon, Littlejohn, Wellman, Yoon)**

\[ \ell^n_{\alpha,\beta}[y] = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=1}^{n} \binom{n}{j}_{\alpha,\beta} (-1)^j \left( (1 - x)^{\alpha+j} (1 + x)^{\beta+j} y^{(j)} \right)^{(j)} \]
\[ \ell_{\alpha,\beta}[y] = -(1 - x^2)y'' + (\alpha - \beta + (\alpha + \beta + 2)x)y' \]

**Theorem (Everitt, Kwon, Littlejohn, Wellman, Yoon)**

\[ \ell^n_{\alpha,\beta}[y] = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=1}^{n} \binom{n}{j}_{\alpha,\beta} (-1)^j \left( (1 - x)^{\alpha+j}(1 + x)^{\beta+j}y^{(j)} \right)^{(j)} \]

\[ \binom{n}{j}_{\alpha,\beta} = \binom{n-1}{j-1}_{\alpha,\beta} + j(j + \alpha + \beta + 1) \binom{n-1}{j}_{\alpha,\beta} \]
What Does $\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha,\beta}$ Count?

$$2\gamma - 1 = \alpha + \beta + 1$$

$$[n]_2 := \{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$$
What Does $\{\binom{n}{j}\}_{\alpha,\beta}$ Count?

$$2\gamma - 1 = \alpha + \beta + 1$$

$$[n]_2 := \{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$$

Theorem (AEGL)

For any positive integer $\gamma$, the Jacobi-Stirling number $\binom{n}{j}_\gamma$ counts set partitions of $[n]_2$ into $j + \gamma$ blocks such that

1. There are $\gamma$ distinguishable zero blocks, any of which may be empty.
2. There are $j$ indistinguishable nonzero blocks, all nonempty.
3. The union of the zero blocks does not contain both copies of any number.
4. Each nonzero block contains both copies of its smallest element and does not contain both copies of any other number.

These are Jacobi-Stirling set partitions.
$$z = \alpha + \beta$$
A Generating Function Interpretation

\[ z = \alpha + \beta \]

\[ S(n, j) := \text{Jacobi-Stirling set partitions of } [n]_2 \]
\[ \text{into 1 zero block and } j \text{ nonzero blocks.} \]
A Generating Function Interpretation

\[ z = \alpha + \beta \]

\[ S(n,j) := \text{Jacobi-Stirling set partitions of } [n]_2 \]
\[ \text{into 1 zero block and } j \text{ nonzero blocks.} \]

**Theorem (Gelineau, Zeng)**

The Jacobi-Stirling number \( \binom{n}{j}_z \) is the generating function in \( z \) for \( S(n,j) \) with respect to the number of numbers with subscript 1 in the zero block.
A Generating Function Interpretation

\[ z = \alpha + \beta \]

\[ S(n,j) := \text{Jacobi-Stirling set partitions of } [n]_2 \text{ into 1 zero block and } j \text{ nonzero blocks.} \]

**Theorem (Gelineau, Zeng)**

The Jacobi-Stirling number \( \left\{ \begin{array}{c} n \\ j \end{array} \right\}_z \) is the generating function in \( z \) for \( S(n,j) \) with respect to the number of numbers with subscript 1 in the zero block.

**Corollary**

The leading coefficient in \( \left\{ \begin{array}{c} n \\ j \end{array} \right\}_z \) is the Stirling number \( \left\{ \begin{array}{c} n \\ j \end{array} \right\} \).
Jacobi-Stirling Numbers of the First Kind

\[ x^n = \sum_{j=0}^{n} \binom{n}{j} \prod_{k=0}^{j-1} (x - k(k + \alpha + \beta + 1)) \]
Theorem (AEGL)

$(-1)^{-j-n}$

The Jacobi-Stirling Numbers of the First Kind

$x^n = \sum_{j=0}^{n} \binom{n}{j} \prod_{k=0}^{j-1} (x - k(k + \alpha + \beta + 1))$

$\prod_{k=0}^{n-1} (x - k(k + \alpha + \beta + 1)) = \sum_{j=0}^{n} (-1)^{n+j} \left[ \begin{array}{c} n \\ j \end{array} \right]_{\alpha,\beta} x^j$
Jacobi-Stirling Numbers of the First Kind

\[ x^n = \sum_{j=0}^{n} \binom{n}{j}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k + \alpha + \beta + 1)) \]

\[ \prod_{k=0}^{n-1} (x - k(k + \alpha + \beta + 1)) = \sum_{j=0}^{n} (-1)^{n+j} \left[ \begin{array}{c} n \\ j \end{array} \right]_{\alpha,\beta} x^j \]

\[ \left[ \begin{array}{c} n \\ j \end{array} \right]_{\alpha,\beta} = \left[ \begin{array}{c} n - 1 \\ j - 1 \end{array} \right]_{\alpha,\beta} + (n - 1)(n + \alpha + \beta) \left[ \begin{array}{c} n - 1 \\ j \end{array} \right]_{\alpha,\beta} \]
Jacobi-Stirling Numbers of the First Kind

\[
x^n = \sum_{j=0}^{n} \binom{n}{j}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k + \alpha + \beta + 1))
\]

\[
\prod_{k=0}^{n-1} (x - k(k + \alpha + \beta + 1)) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j}_{\alpha,\beta} x^j
\]

\[
\begin{align*}
\binom{n}{j}_{\alpha,\beta} &= \binom{n-1}{j-1}_{\alpha,\beta} + (n-1)(n+\alpha+\beta) \binom{n-1}{j}_{\alpha,\beta}
\end{align*}
\]

Theorem (AEGL)

\[
\begin{align*}
\binom{n}{-j}_{\gamma} &= (-1)^{n+j} \binom{n}{j}_{1-\gamma}
\end{align*}
\]
Theorem (AEGL)

For any positive integer $\gamma$, the Jacobi-Stirling number of the first kind $\genfrac{[}{]}{0pt}{}{n}{j}_\gamma$ is the number of ordered pairs $(\pi_1, \pi_2)$ of permutations with $\pi_1 \in S_{n+\gamma}$ and $\pi_2 \in S_{n+\gamma-1}$ such that

1. $\pi_1$ has $\gamma + j$ cycles and $\pi_2$ has $\gamma + j - 1$ cycles.
2. The cycle maxima of $\pi_1$ which are less than $n + \gamma$ are exactly the cycle maxima of $\pi_2$.
3. For each non cycle maximum $k$, at least one of $\pi_1(k)$ and $\pi_2(k)$ is less than or equal to $n$.

Such ordered pairs are balanced Jacobi-Stirling permutations.
If $2\gamma$ is a positive integer, then the Jacobi-Stirling number of the first kind \( \left[ \begin{array}{c} n \\ j \end{array} \right] _{\gamma} \) is the number of ordered pairs \((\pi_1, \pi_2)\) of permutations with 

\[ \pi_1 \in S_{n+\gamma} \text{ and } \pi_2 \in S_n \text{ such that} \]

1. \( \pi_1 \) has \( j + 2\gamma - 1 \) cycles and \( \pi_2 \) has \( j \) cycles.
2. The cycle maxima of \( \pi_1 \) which are less than \( n + 1 \) are exactly the cycle maxima of \( \pi_2 \).
3. For each non cycle maximum \( k \), at least one of \( \pi_1(k) \) and \( \pi_2(k) \) is less than or equal to \( n \).

Such ordered pairs are \textit{unbalanced Jacobi-Stirling permutations}. 
\[ \Sigma(n, j) := \text{all } (\sigma, \tau) \text{ such that} \]

- $\sigma$ is a permutation of \{0, 1, \ldots, n\}, $\tau$ is a permutation of \{1, 2, \ldots, n\}, and both have $j$ cycles.
- 1 and 0 are in the same cycle in $\sigma$.
- Among their nonzero entries, $\sigma$ and $\tau$ have the same cycle minima.
\[ \Sigma(n, j) := \text{all } (\sigma, \tau) \text{ such that} \]
- \( \sigma \) is a permutation of \( \{0, 1, \ldots, n\} \), \( \tau \) is a permutation of \( \{1, 2, \ldots, n\} \), and both have \( j \) cycles.
- 1 and 0 are in the same cycle in \( \sigma \).
- Among their nonzero entries, \( \sigma \) and \( \tau \) have the same cycle minima.

**Theorem (Gelineau, Zeng)**

\[ \left[ \begin{array}{c} n \\ j \end{array} \right]_z \text{ is the generating function in } z \text{ for } \Sigma(n, j) \text{ with respect to the number of nonzero left-to-right minima in the cycle containing 0 in } \sigma, \text{ written as a word beginning with } \sigma(0). \]
Where are the Symmetric Functions?

\[ h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j) \]
\[ h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j) \]

\[ \left\{ \binom{n}{j} \right\}_z = h_{n-j}(1(1 + z), 2(2 + z), \ldots, j(j + z)) \]
Where are the Symmetric Functions?

\[ h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j) \]

\[ \begin{bmatrix} n \\ j \end{bmatrix}_z = h_{n-j}(1(1 + z), 2(2 + z), \ldots, j(j + z)) \]

\[ e_{n-j}(x_1, \ldots, x_{n-1}) = e_{n-j}(x_1, \ldots, x_{n-2}) + x_{n-1} e_{n-j-1}(x_1, \ldots, x_{n-2}) \]
Where are the Symmetric Functions?

\[ h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j) \]

\[ \begin{bmatrix} n \\ j \end{bmatrix}_z = h_{n-j}(1(1 + z), 2(2 + z), \ldots, j(j + z)) \]

\[ e_{n-j}(x_1, \ldots, x_{n-1}) = e_{n-j}(x_1, \ldots, x_{n-2}) + x_{n-1} e_{n-j-1}(x_1, \ldots, x_{n-2}) \]

\[ \begin{bmatrix} n \\ j \end{bmatrix}_z = e_{n-j}(1(1 + z), 2(2 + z), \ldots, (n - 1)(n - 1 + z)) \]
An Open \( q \)-uestion

Is there a \( q \)-analogue of the Jacobi-Stirling numbers associated with the \( q \)-Jacobi polynomials?
Thank You!