Functions on Young’s Lattice

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Abstract

We study a family of functions on Young’s lattice related to certain interesting families of Catalan paths. We conjecture that for all of these functions, the value on any shape is the sum of the values on the children of that shape. We prove this conjecture for shapes with at most 5 rows, and for shapes which are “wide” in a certain technical sense. We also prove this conjecture for certain specializations of these functions.

1 Introduction

A partition of \( n \) is a way to write \( n \) as a sum of positive integers, written in nonincreasing order. For instance, the partitions of 5 are: 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1. Once you have a partition, you can draw its diagram by drawing the number of boxes that corresponds with each consecutive value of the partition. For instance, draw four boxes, then on top of those draw three boxes, then two boxes for the partition 4,3,2. In Figure 1, the partition diagram of 6,5,3,2,2 is shown.

\[ \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & & \\
\cdot & & & \\
\cdot & & & \\
\end{array} \]

Figure 1: The partition diagram of 6,5,3,2,2.

The children of a partition \( \lambda \) are the partitions built by adding a single box to the diagram of \( \lambda \). Young’s lattice is the partially ordered set in which each partition is covered by its children. (For more information on partitions, see [1].)

A diagram of a partition can be decomposed by drawing in the border strips, which are groups of lines and points that follow the northeast border of the shape. To decompose a partition into border strips we repeatedly form border strips until every box is part of a border strip. We label each row with the number of boxes in the border strip which starts in that row. Then \( a_i \) is the
label of the $i$th row, numbering from top to bottom. If no border strip starts in row $i$, then $a_i = 0$. In Figure 2, $a_1 = 10$ and $a_5 = 1$.

In Figure 2, $a_1 = 10$ and $a_5 = 1$.

![Figure 2: The border strip decomposition of 6,5,3,2,2.](image)

We compute the sign of each shape $s_\lambda$ by counting the horizontal lines that get crossed in its border strip decomposition. If the number of horizontal lines that get crossed is even, then the sign of $\lambda$ is 1, and if it’s odd then the sign of $\lambda$ is $-1$. For example, the partition 6,5,3,2,2 has 7 crossings so its sign is $-1$.

For each partition $\lambda$ we are interested in a certain polynomial in the variables $X_0, X_1, \ldots$, though we almost always take $X_0 = X_1 = 1$. To obtain this polynomial, first suppose $\lambda$ has $k$ parts and the border strip decomposition of $\lambda$ gives us row labels $a_1, \ldots, a_k$. Now let $M_\lambda$ be the $k \times k$ matrix whose $ij$th entry is $X_{a_j-i+j}$; here $X_0 = X_1 = 1$ and $X_n = 0$ for $n < 0$. Up to a sign, we are interested in the polynomial which is the determinant of $M_\lambda$. In particular, for any partition $\lambda$, let $M_\lambda$ be the matrix described above, and let $s(\lambda)$ be the sign obtained from the border strip decomposition of $\lambda$. Then we write $G_\lambda(1,1,X_2,\ldots)$ to denote the polynomial given by

$$G_\lambda = s(\lambda) \det(M_\lambda).$$

When $X_n = C_n = \frac{1}{n+1} \binom{2n}{n}$ for all $n \geq 0$, the polynomial $G_\lambda(X_0,X_1,\ldots)$ has a combinatorial interpretation in terms of nests of nonintersecting lattice paths. To describe this interpretation, recall that a Catalan path is a path of horizontal and vertical lines connecting consecutive lattice points that begins and ends on the line $y = x$, and does not pass below this line. A nest of nonintersecting lattice paths is a collection of Catalan paths in which no two paths intersect. For any partition $\lambda$, the quantity $G_\lambda(C_0,C_1,\ldots)$ is, up to a sign, the number of lists $\pi_1, \ldots, \pi_k$ of nonintersecting Catalan paths such that the following hold.

- $\pi_i$ is strictly above $\pi_{i+1}$ for all $i$.
- $\pi_i$ begins at $(i-1,i-1)$ and ends at $(i+a_i-1,i+a_i-1)$.

This follows by techniques of Gessel and Viennot [2], and these nests of paths are related to the Bergeron-Garsia \(\triangledown\) operator [3].

It turns out that the value of $G_\lambda$ is often closely related to the values of $G_{\mu_1}, \ldots, G_{\mu_k}$, where $\mu_1, \ldots, \mu_k$ are the children of $\lambda$. To capture this relationship, we make the following Definition.

**Definition 1.** Suppose $\lambda$ is a partition and $\mu_1, \ldots, \mu_k$ are the children of $\lambda$. We say $\lambda$ is familial with respect to the sequence $\{X_i\}_{i=0}^\infty$ whenever

$$G_\lambda(X_0,X_1,\ldots) = \sum_{i=1}^k G_{\mu_i}(X_0,X_1,\ldots).$$

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The following conjecture is the primary point of the paper.

**Conjecture 2.** Every partition $\lambda$ is familial with respect to any sequence $1, 1, X_2, X_3, \ldots$.

Although one can use representations of the symmetric group to prove this conjecture, we prove directly that this conjecture holds for all shapes with at most 5 rows, and for a certain class of wide shapes. We also investigate this conjecture for several specializations of $\{X_i\}_{i=2}^\infty$.

## 2 Short Shapes

In this section we show that Conjecture 1 holds for short shapes, which are shapes with at most 5 rows.

**Proposition 3.** If $\lambda$ has exactly one row then $\lambda$ is familial with respect to $1, 1, X_2, X_3, \ldots$.

**Proof.** If $\lambda = m$ then

$$G_{\lambda} = \det \left( \begin{array}{c} X_m \end{array} \right) = X_m.$$

If $m > 1$, then the children of $\lambda$ are $\mu_1 = m, 1$ and $\mu_2 = m + 1$. For these children we have

$$G_{\mu_1} = X_m - X_{m+1}$$

and

$$G_{\mu_2} = X_{m+1}.$$

Now it is routine to check that $G_{\lambda} = G_{\mu_1} + G_{\mu_2}$. \qed

**Proposition 4.** If $\lambda$ has exactly two rows then $\lambda$ is familial with respect to $1, 1, X_2, X_3, \ldots$.

**Proof.** If $\lambda = m, k$ then

$$G_{\lambda} = s(\lambda) \det \left( \begin{array}{cc} X_{m+1} & X_k \\ X_m & X_{k-1} \end{array} \right),$$

where $s(\lambda)$ is the sign of $\lambda$. If $m > k$, then the children of $\lambda$ are $\mu_1 = m, k, 1$, $\mu_2 = m, k + 1$, and $\mu_3 = m + 1, k$. For these children we have

$$G_{\mu_1} = X_{m+2}X_{k-1} - X_{m+1}X_{k-1} - X_mX_{k+1} + X_mX_k,$$

$$G_{\mu_2} = X_mX_{k+1} - X_{m+1}X_k,$$

and

$$G_{\mu_3} = X_{m+1}X_k - X_{m+2}X_{k-1}.$$

Now it is routine to check that $G_{\lambda} = G_{\mu_1} + G_{\mu_2} + G_{\mu_3}$. The proof when $m = k$ is similar. \qed
Proposition 5. If $\lambda$ has exactly three rows then $\lambda$ is familial with respect to $1,1,X_2,X_3,\ldots$.

Proof. If $\lambda = m,k,n$ then

$$G_{\lambda} = s(\lambda) \det \begin{pmatrix} X_{m+2} & X_{k+1} & X_n \\ X_{m+1} & X_k & X_{n-1} \\ X_m & X_{k-1} & X_{n-2} \end{pmatrix}.$$ 

If $m > k$, then the children of $\lambda$ are $\mu_1 = m,k,n,1,\mu_2 = m,k,n+1,\mu_3 = m,k+1,n$, and $\mu_4 = m+1,k,n$. For these children we have

$$G_{\mu_1} = X_{m+1}X_{k+1}X_{n-2} - X_{m+1}X_{k+2}X_{n-2} - X_{m+2}X_kX_{n-2} + X_{m+3}X_kX_{n-2} - X_mX_{k+1}X_{n-1} + X_mX_{k+2}X_{n-1} + X_{m+2}X_{k-1}X_{n-1} - X_{m+3}X_{k-1}X_{n-1} + X_mX_kX_{n-1} - X_{m+1}X_{k-1}X_{n+1},$$

$$G_{\mu_2} = X_{m+1}X_{k+1}X_{n-1} - X_{m+2}X_kX_{n-1} - X_mX_{k+1}X_n + X_{m+2}X_{k-1}X_n + X_mX_kX_{n+1} - X_{m+1}X_{k-1}X_{n+1},$$

$$G_{\mu_3} = X_{m+1}X_{k+2}X_{n-2} - X_{m+2}X_{k+1}X_{n-2} - X_{m+3}X_kX_{n-2} + X_{m+3}X_kX_{n-2} + X_mX_kX_{n-1} - X_{m+1}X_{k-1}X_n,$$

and

$$G_{\mu_4} = X_{m+2}X_{k+1}X_{n-2} - X_{m+3}X_kX_{n-2} - X_{m+1}X_{k+1}X_{n-1} + X_{m+3}X_{k-1}X_{n-1} + X_{m+1}X_kX_n - X_{m+2}X_{k-1}X_n.$$

Now it is routine to check that $G_{\lambda} = G_{\mu_1} + G_{\mu_2} + G_{\mu_3} + G_{\mu_4}$. The proof when $m = k$ is similar.

Proposition 6. If $\lambda$ has exactly four rows then $\lambda$ is familial with respect to $1,1,X_2,X_3,\ldots$.

Proof. This is similar to the proof of Proposition 5.

Proposition 7. If $\lambda$ has exactly five rows then $\lambda$ is familial with respect to $1,1,X_2,X_3,\ldots$.

Proof. This is similar to the proof of Proposition 5.

3 Wide Shapes

Notice that each partition has a child whose top row has only one box. In addition, this child has a portion of its determinant that will always cancel with the parent determinant. In this section we show that if the shape of our partition is wide enough, then the children that fall below the diagonal will all cancel with each other, and our conjecture will hold in this case.

Definition 8. For any numbers $i_1,\ldots,i_n$, set

$$A(i_1,i_2,\ldots,i_n) = \begin{pmatrix} X_{i_1} & X_{i_2} & \cdots & X_{i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ X_{i_n-1} & \cdots & \cdots & X_{i_n-1} \end{pmatrix}$$

and
\[ B(i_1, i_2, \ldots, i_n) = \begin{pmatrix}
X_{i_1+1} & X_{i_2+1} & \cdots & X_{i_n+1} \\
X_{i_1-1} & X_{i_2-1} & \cdots & X_{i_n-1} \\
\vdots & \vdots & \ddots & \vdots \\
X_{i_1-n+1} & \cdots & X_{i_n-n+1}
\end{pmatrix}. \]

Here \( X_0 = X_1 = 1 \) and \( X_{-n} = 0 \) for \( n \geq 1 \).

**Lemma 9.** For all \( i_1, \ldots, i_n \) we have

\[
\det(B(i_1, i_2, \ldots, i_n)) - \sum_{j=1}^{n} \det(A(i_1, \ldots, i_j+1, \ldots, i_n)) = 0. \tag{1}
\]

**Proof.** We argue by induction on \( n \). When \( n = 1 \) we have

\[
\det(X_{i_1+1}) - \det(X_{i_1+1}) = 0.
\]

Now suppose \( n > 1 \) and (1) holds for \( n \). The coefficient of \( X_{i_1+1} \) in

\[
\det(B(i_1, \ldots, i_n, i_{n+1})) - \sum_{j=1}^{n+1} \det(A(i_1, \ldots, i_j+1, \ldots, i_n, i_{n+1}))
\]

is

\[
\det\begin{pmatrix}
X_{i_2-1} & X_{i_2-1} & \cdots & X_{i_n+1-1} \\
X_{i_2-2} & X_{i_2-2} & \cdots & X_{i_n+1-2} \\
\vdots & \vdots & \ddots & \vdots \\
X_{i_2-n} & X_{i_2-n} & \cdots & X_{i_n+1-n}
\end{pmatrix}
- \det\begin{pmatrix}
X_{i_2-1} & X_{i_2-1} & \cdots & X_{i_n+1-1} \\
X_{i_2-2} & X_{i_2-2} & \cdots & X_{i_n+1-2} \\
\vdots & \vdots & \ddots & \vdots \\
X_{i_2-n} & X_{i_2-n} & \cdots & X_{i_n+1-n}
\end{pmatrix} = 0.
\]

Similarly, the coefficient of \( X_{i_k+1} \) is 0 for \( 1 \leq k \leq n+1 \). The coefficient of \( X_{i_1} \) in the quantity in (2) is

\[
\det(B(i_2 - 1, i_3 - 1, \ldots, i_{n+1} - 1)) - \sum_{j=2}^{n+1} \det(A(i_2 - 1, \ldots, i_j, \ldots, i_{n+1} - 1)),
\]

which is 0 by induction. Similarly, the coefficient of \( X_{i_k} \) is 0 for \( 1 \leq k \leq n+1 \). Since every term in the quantity in (2) has \( X_{i_k} \) or \( X_{i_{k+1}} \) as a factor for some \( k \), the result follows.

**Theorem 10.** If \( \lambda \) has only one child whose new box is above \( y = x \), then \( \lambda \) is familial with respect to \( 1, 1, X_2, X_3, \ldots \).

**Proof.** If one child, \( \mu \), has a single box on top of its diagram, then \( G_\mu \) is

\[
G_\mu = \det\begin{pmatrix}
X_{i_1} & 1 & X_{i_2} & \cdots & X_{i_n} \\
X_{i_1-1} & 1 & X_{i_2-1} & \cdots & X_{i_n-1} \\
0 & 0 & \cdots & \cdots \\
X_{i_1-j} & 0 & \cdots & \cdots
\end{pmatrix}.
\]

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Expand the determinant along the 2nd column. One resulting term is the value of $G_\lambda$, so we need to show

$$0 = \det \begin{pmatrix} X_{i_1} & X_{i_2} & \cdots & X_{i_n} \\ X_{i_1-2} & X_{i_2-2} & \cdots & X_{i_n-2} \\ \vdots & \vdots & \ddots & \vdots \\ X_{i_1-n} & 0 & \cdots & X_{i_n-n} \end{pmatrix} - \sum_{j=1}^n G_{\mu_j},$$

where $\mu_1, \ldots, \mu_n$ are the other children of $\lambda$. However, for each $j$,

$$G_{\mu_j} = \det(A(i_1 - 1, i_2 - 1, \ldots, i_j, \ldots, i_n - 1)).$$

Now the result follows from Lemma 9.

4 Special Sequences

So far we have studied the polynomial $G_\lambda$ for the general sequence $1, 1, X_2, X_3, \ldots$. In this section we study $G_\lambda$ for certain specializations of the $X_i$’s. We begin with the Fibonacci numbers, which are defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

**Proposition 11.** If $\lambda$ has 3 or more rows, then $G_\lambda(F_0, F_1, \ldots) = 0$.

**Proof.** If $\lambda$ has 3 or more rows then

$$G_\lambda(F_0, F_1, \ldots) = s(\lambda) \det \begin{pmatrix} F_{k+2} & \cdots & F_{m+2} & F_{n+2} \\ F_{k+1} & \cdots & F_{m+1} & F_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ F_k & \cdots & F_m & F_n \end{pmatrix}.$$

To show this determinant is 0, suppose there are numbers $x$ and $y$ so that

$$\begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix} = x \begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix} + y \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}. \quad (3)$$

Then

$$xF_{m+2} + yF_{n+2} = xF_{m+1} + yF_{n+1} + xF_m + yF_n = F_{k+1} + F_k = F_{k+2}.$$ 

Repeating this, we see that the first column is a linear combination of the last two, so the determinant is 0, as claimed.

If there are not numbers $x$ and $y$ so that (3) holds then there are numbers $x$ and $y$ so that

$$x \begin{pmatrix} F_{m+1} \\ F_m \end{pmatrix} + y \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now a similar argument shows that the last two columns are linearly dependent, so the determinant is 0. \qed
\textbf{Corollary 12.} Every partition is familial with respect to the Fibonacci numbers.

\textit{Proof.} This is immediate from Proposition 11 and Propositions 3 and 4. \hfill \Box

We now consider $G_\lambda$ for the tribonacci numbers, which are defined by $T_0 = T_1 = 1, T_2 = 2,$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3.$

\textbf{Proposition 13.} If $\lambda$ has 4 or more rows, then $G_\lambda(T_0, T_1, \ldots) = 0.$

\textit{Proof.} If $\lambda$ has 4 or more rows then

$$G_\lambda(T_0, T_1, \ldots) = s(\lambda) \det \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ T_{d+2} & \ldots & T_{a+2} & T_{b+2} & T_{c+2} \\ T_{d+1} & \ldots & T_{a+1} & T_{b+1} & T_{c+1} \\ T_d & \ldots & T_a & T_b & T_c \end{pmatrix}.$$ 

Arguing as in the proof of Proposition 11, we see that the four given columns of this matrix are linearly dependent, so the determinant is 0. \hfill \Box

\textbf{Corollary 14.} Every partition is familial with respect to the tribonacci numbers.

\textit{Proof.} This is immediate from Proposition 13 and Propositions 3, 4, and 5. \hfill \Box

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\textbf{References}

