The Generalized Terwilliger Algebra of a Finite Group

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Notation

$G :=$ finite group

$\mathbb{C}G :=$ complex group algebra of G

$\mathbb{C}[G] :=$ algebra of functions from G to $\mathbb{C}$

$Z :=$ center of $\mathbb{C}G$

$C :=$ functions from G to $\mathbb{C}$
which are
constant on conjugacy classes

The Terwilliger Algebra of G

- action of Z on $\mathbb{C}G$: multiplication
  - faithful

- action of C on $\mathbb{C}G$: $f \cdot g = f(g)g$
  - faithful

- action of G on $\mathbb{C}G$: conjugation
  $$g \cdot x = g^{-1}xg$$

- actions of Z and C commute with action of G

$$\text{view} \begin{cases} Z \subseteq End_G(\mathbb{C}G) \\ \\ C \subseteq End_G(\mathbb{C}G) \end{cases}$$

$T :=$ subalgebra of $End_G(\mathbb{C})$ generated by Z and C

Bases for Z

$C_0 := \{e\}, C_1, \ldots, C_d$ conjugacy classes of G

Define $X_i = \sum_{g \in C_i} g$

$X_0, \ldots, X_d$ is a basis for Z

Z is commutative and semisimple, so it has a

basis $E_0, \ldots, E_d$

$E_i E_j = \delta_{ij} E_i$

$$\sum_{i=0}^{d} E_i = I$$

$$E_0 = |G|^{-1} \sum_{i=0}^{d} X_i$$

Bases for C

$\rho : \mathbb{C}[G] \longrightarrow \mathbb{C}G$ linear

$$\rho(f) = \sum_{g \in G} f(g) g$$

$$\rho \text{ is } \begin{cases} \text{a vector space isomorphism} \\ \text{not a } \mathbb{C}\text{-algebra isomorphism} \end{cases}$$

$$\rho(C) = Z$$

$$X_i^* = \rho^{-1}(|G|^{-1} E_i)$$

$$E_i^* = \rho^{-1}(X_i)$$

$$E_i^* E_j^* = \delta_{ij} E_i^*$$

$$\sum_{i=0}^{d} E_i^* = I$$

$$E_0^* = |G|^{-1} \sum_{i=0}^{d} X_i^*$$

Terwilliger's Triple Product Relations

Define p_{ij}^{h} by

$$X_i X_j = \sum_{h=0}^{d} p_{ij}^{h} X_h$$

Define p_{ij}^{h*} by

$$X_i^* X_j^* = \sum_{h=0}^{d} p_{ij}^{h*} X_h^*$$

$E_h^* X_i E_j^* = 0$ if and only if $p_{ij}^{h} = 0$

$E_h X_i^* E_j = 0$ if and only if $p_{ij}^{h*} = 0$

To what extent do these relations determine T's structure?

The Generalized Terwilliger Algebra $\mathcal{T}$ of G

Generators

$x_0, \ldots, x_d$ $\qquad$ $x_0^*, \ldots, x_d^*$

Relations

$$x_0 = x_0^*$$

$$x_i x_j = \sum_{h=0}^{d} p_{ij}^h x_h \qquad x_i^* x_j^* = \sum_{h=0}^{d} p_{ij}^{h*} x_h^*$$

$e_h^* x_i e_j^* = 0$ if $p_{ij}^h = 0$ $\qquad$ $e_h x_i^* e_j = 0$ if $p_{ij}^{h*} = 0$

Notation

$$e_i = |G|^{-1} d_i \sum_{j=0}^{d} \overline{\chi_i(g_j)} x_j \qquad e_i^* = |G|^{-1} |C_i| \sum_{j=0}^{d} d_j^{-1} \chi_j(g_i) x_j^*$$

$\chi_i :=$ irreducible character for E_i $\qquad$ $d_i := \deg \chi_i$

$C_i := i$th conjugacy class $\qquad$ $g_i \in C_i$

Facts About $\mathcal{T}$

$\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_1$ (direct sum)

$\mathcal{T}_0$ is $\mathbb{C}$-algebra isomorphic to $M_{d+1}(\mathbb{C})$

$\mathbb{C}$-algebra homomorphism:

$$\begin{array}{ccc} & \dagger & \\ \mathcal{T} & \longrightarrow & T \\ x_i & \mapsto & X_i \\ x_i^* & \mapsto & X_i^* \\ e_i & \mapsto & E_i \\ e_i^* & \mapsto & E_i^* \end{array}$$

Action of $\mathcal{T}$ on $\mathbb{C}G$:

$$x \cdot h := \dagger(x) \cdot h$$

$$\mathcal{T}_0 \, \mathbb{C}G = Z$$

For Which G is $\mathcal{T}(G)$ Simple?

Theorem (E)

$\mathcal{T}$ is simple if and only if G is Abelian.

In this case, $\mathcal{T} \cong T \cong M_{d+1}(\mathbb{C})$.

The Product Subspaces of $\mathcal{T}$

$$\mathcal{Z} := Span\{x_0, \ldots, x_d\} \qquad \mathcal{C} := Span\{x_0^*, \ldots, x_d^*\}$$

Theorem (E)

$$\dim \mathcal{C}\mathcal{Z} = (d+1)^2 = \dim \mathcal{Z}\mathcal{C}$$

Theorem (E)

The following are equivalent.

(i) $\mathcal{C}\mathcal{Z} = \mathcal{Z}\mathcal{C}$.

(ii) $\mathcal{T} = \mathcal{C}\mathcal{Z}$.

(iii) $\mathcal{T} = \mathcal{Z}\mathcal{C}$.

(iv) G is Abelian.

When these hold we have $x_j^* x_i = \chi_j(g_i) x_i x_j^*$.

Open Questions

1. For which groups do we have $\mathcal{T} \cong T$?

2. For which groups is $\mathcal{T}$ finite-dimensional?

3. If $\mathcal{T}$ is finite-dimensional, must we have $\mathcal{T} \cong T$?

4. For which groups is $\mathcal{T}$ semisimple?

5. If $\mathcal{T}$ is semisimple, must we have $\mathcal{T} \cong T$?

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