A Chromatic Symmetric Function for Signed Graphs

Eric S. Egge

Carleton College

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Our Graphs

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Not Interesting
A **proper coloring** of a graph $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices have different colors.
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Proper Coloring

Not a Proper Coloring
Our “colors” are the variables $x_1, x_2, x_3, \ldots$.
The Chromatic Symmetric Function of a Graph

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For any proper coloring $C$ of $G$, $x(C)$ is the product of the colors.
The Chromatic Symmetric Function of a Graph

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For any proper coloring $C$ of $G$, $x(C)$ is the product of the colors.

Definition (Stanley)
The chromatic symmetric function of $G$ is

$$X_G = \sum_{C \text{ proper coloring of } G} x(C).$$
Signed Graphs

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A signed graph is a graph in which every edge is given a sign, either + or -. 
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Definition
A signed graph is a graph in which every edge is given a sign, either $+$ or $-$.
In a signed graph with sign function $\sigma$, assign a sign $S(v)$ to each vertex $v$. 
Switching

In a signed graph with sign function $\sigma$, assign a sign $S(v)$ to each vertex $v$.

If $e$ connects $v_1$ and $v_2$ then we get a new sign function $\tau$ on edges

$$\tau(e) = S(v_1)\sigma(e)S(v_2)$$
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Proper Colorings of Signed Graphs

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A proper coloring of a signed graph is a coloring in which

\[ x_a \neq x_{\sigma b} \]

\textbf{Fact}

If \( G \) and \( H \) are related by switching then there is a natural bijection between their sets of proper colorings.
Proper Colorings of Signed Graphs

A proper coloring of a signed graph is a coloring in which

\[ \chi_a \sim_{\sigma} \chi_b \]

implies

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Fact

If \( G \) and \( H \) are related by switching then there is a natural bijection between their sets of proper colorings.
The Chromatic Symmetric Function of a Signed Graph

Definition
For a signed graph $G$, the chromatic symmetric function of $G$ is

$$Y_G = \sum_{C \text{ proper coloring of } G} x(C).$$
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For a signed graph $G$, the chromatic symmetric function of $G$ is

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Observation
$Y_G$ is invariant under the natural action of the hyperoctahedral group, which is the set of permutations $\pi$ of $\pm 1, \pm 2, \ldots$ such that

$$\pi(-j) = -\pi(j)$$

for all $j$. 
The Chromatic Symmetric Function of a Signed Graph

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Observation

$$Y_G \in BSym$$
Marked Ferrers Diagrams

Goal: a basis for $BSym$. 

Definition

A marked Ferrers diagram is a Ferrers diagram in which some (or no) boxes contain dots, such that

▶ the rows of dotted boxes are left-justified and
▶ for each $k$, the dotted boxes in the rows of length $k$ form a Ferrers diagram.

$|\lambda| =$ total number of boxes and dots in $\lambda$.
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Marked Ferrers Diagrams and Their Monomials

For each marked Ferrers diagram there is a monomial.
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\[ x_1 x_{-1} x_2 x_3 x_4 x_5 \]

\[ x_6^3 x_2^3 x_4^5 x_{-4}^4 \cdots \]

\[ x_1^7 x_{-1}^2 \]
A \textit{BSym} Basis

\[ BSym_n := \text{space of homogeneous invariant series of total degree } n \]
A $BSym$ Basis

$BSym_n :=$ space of homogeneous invariant series of total degree $n$

For any marked Ferrers diagram $\lambda$, $m_\lambda$ is the sum of the distinct images of $\lambda$’s monomial.
A $\text{BSym}$ Basis

$\text{BSym}_n := \text{space of homogeneous invariant series of total degree } n$

For any marked Ferrers diagram $\lambda$, $m_\lambda$ is the sum of the distinct images of $\lambda$’s monomial.

**Theorem**

$\{m_\lambda \mid |\lambda| = n\}$ is a basis for $\text{BSym}_n$. 
\[ \dim BSym_n \]

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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>6</th>
<th>7</th>
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Theorem

$$\sum_{n=0}^{\infty} \dim(B\text{Sym}_n)x^n = \prod_{j=1}^{\infty} \left( \frac{1}{1 - x^j} \right)^{\lfloor j/2 \rfloor + 1}$$
The Power Sum Basis

\[ p_\lambda := m_\lambda \text{ for any } \lambda \text{ with just one row} \]
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\[ p_{\lambda_1, \ldots, \lambda_k} := p_{\lambda_1} \cdots p_{\lambda_k} \]

for any list \( \lambda_1, \ldots, \lambda_k \) of row shapes
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**Theorem**

*If we linearly order the set of row shapes then*

\[ \{ p_{\lambda_1, \ldots, \lambda_k} \mid \sum_j |\lambda_j| = n \text{ and } \lambda_1 \geq \cdots \geq \lambda_k \} \]

*is a basis for \( BSym_n \).*
The Elementary Basis?

\[ e_\lambda := m_\lambda \text{ for any } \lambda \text{ with just one column} \]

\[ e_{\lambda_1, \ldots, \lambda_k} := e_{\lambda_1} \cdots e_{\lambda_k} \]

for any list \( \lambda_1, \ldots, \lambda_k \) of column shapes

Conjecture

If we linearly order the set of column shapes then

\[ \{ e_{\lambda_1, \ldots, \lambda_k} \mid \sum_j |\lambda_j| = n \text{ and } \lambda_1 \geq \cdots \geq \lambda_k \} \]

is a basis for \( BSym_n \).
Basic Results: The Chromatic Polynomial

Definition
The chromatic polynomial \( \chi_G(n) \) of a signed graph \( G \) is the number of proper colorings of \( G \) with \( x_1, x_{-1}, \ldots, x_n, x_{-n} \).
Basic Results: The Chromatic Polynomial

Definition
The chromatic polynomial $\chi_G(n)$ of a signed graph $G$ is the number of proper colorings of $G$ with $x_1, x_{-1}, \ldots, x_n, x_{-n}$.

Theorem
If $G$ is a signed graph then

$$Y_G(1, 1, \ldots, 1, 0, 0, \ldots) = \chi_G(n)$$
Basic Results

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If a signed graph $G$ is a disjoint union of signed graphs $G_1$ and $G_2$ then

$$Y_G = Y_{G_1} \cdot Y_{G_2}.$$
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Theorem
If all of the edges in a signed graph $G$ are positive then

$$Y_G = X_G(x_1, x_{-1}, x_2, x_{-2}, \ldots).$$
Switching Does Not Preserve $Y_G$

$m\bullet + 2m\square$  

$m\square + 2m\square$
The Power Basis Expansion

Definition
For any connected, signed graph $G$, the type $\lambda(G)$ of $G$ is the row shape consisting of $k$ boxes and $m$ dots, where $G$ can be colored with $k \times 1$s and $m \times -1$s so that every edge is improper. If $G$ is not connected then its type is the sequence of types of its connected components.
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\[
\lambda(G) = \begin{array}{|c|c|}
\hline
\bullet & \bullet \\
\hline
\end{array}
\]
The Power Basis Expansion

Definition
A connected signed graph $G$ is 2-faced whenever there are two colorings of its vertices with $x_1$ and $x_{-1}$ which are improper along every edge, and which have at least as many $x_1$s as $x_{-1}$s.
The Power Basis Expansion

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A connected signed graph $G$ is **2-faced** whenever there are two colorings of its vertices with $x_1$ and $x_{-1}$ which are improper along every edge, and which have at least as many $x_1$s as $x_{-1}$s.

**Example**
Every path with an even number of vertices is 2-faced.
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Example
Every path with an even number of vertices is 2-faced.

Example
Every cycle with an even number of vertices whose product of signs is positive is 2-faced.
The Power Basis Expansion

Definition
A connected signed graph $G$ is 2-faced whenever there are two colorings of its vertices with $x_1$ and $x_{-1}$ which are improper along every edge, and which have at least as many $x_1$s as $x_{-1}$s.

Theorem
For any signed graph $G$ with edge set $E$, 

$$Y_G = \sum_{S \subseteq E} (-1)^{|S|} 2^{tf(S)} p_{\lambda(S)},$$

where $tf(S)$ is the number of 2-faces of $S$ and $p_{\lambda(S)} = 0$ if $S$ has no type.
The Last Slide

Thank you!