Linear Recurrences and the Pfaffian Transform

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Functions on Sequences

The Binomial Transform

\[
B(a_j)_n = \sum_{j=0}^{n} \binom{n}{j} a_j
\]
Functions on Sequences

The Binomial Transform

\[ B(\{a_j\})_n = \sum_{j=0}^{n} \binom{n}{j} a_j \]

The Hankel Transform

\[ H(\{a_j\})_n = \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & \cdots & a_{n+1} \\ a_2 & \cdots & \cdots & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{pmatrix} \]
Functions on Sequences

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The Hankel Transform

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Theorem (Layman, 2001)

\[ H(B(\{a_j\})) = H(\{a_j\}) \]
The Pfaffian of a Skew-Symmetric Matrix

\[
\begin{vmatrix}
  a & b \\
  c & d \\
\end{vmatrix} = ad - bc
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\text{det} A = \sum_{\pi \in \text{perm} K_{n,n}} (-1)^{\text{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}
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\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc
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\[\det A = \sum_{\pi \in \text{pm} \ K_{n,n}} (-1)^{\text{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}\]

\[\Pf A = \sum_{\pi \in \text{pm} \ K_{2n}} (-1)^{\text{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}\]
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\det A = \sum_{\pi \text{ pm } K_{n,n}} (-1)^{\text{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}
\]

\[
Pf A = \sum_{\pi \text{ pm } K_{2n}} (-1)^{\text{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}
\]

**Fact:** \( \det(A) = (\text{Pf}(A))^2 \)
The Pfaffian Transform

\[
Pf(\{a_j\})_n = Pf \begin{pmatrix}
  0 & a_1 & a_2 & a_3 & \cdots & a_{2n-1} \\
  -a_1 & 0 & a_1 & a_2 & \cdots & a_{2n-2} \\
  -a_2 & -a_1 & 0 & a_1 & \cdots & a_{2n-3} \\
  -a_3 & -a_2 & -a_1 & 0 & \cdots & a_{2n-4} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & -a_{2n-4} & \cdots & 0
\end{pmatrix}
\]
## Pfaffian Transform Examples

<table>
<thead>
<tr>
<th>( {a_j} )</th>
<th>( \text{Pf}({a_j}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, 4, \ldots, 2^{n-1}, \ldots</td>
<td>1, 1, 1, \ldots</td>
</tr>
<tr>
<td>1, 3, 9, \ldots, 3^{n-1}, \ldots</td>
<td>1, 1, 1, \ldots</td>
</tr>
<tr>
<td>1, 1, 2, 3, \ldots, (F_n), \ldots</td>
<td>1, 2, 4, \ldots, 2^{n-1}, \ldots</td>
</tr>
<tr>
<td>1, 1, 3, 5, 11, 21, \ldots, (J_n), \ldots</td>
<td>1, 3, 9, \ldots, 3^{n-1}, \ldots</td>
</tr>
<tr>
<td>1, 1, 2, 4, 7, 13, \ldots, (T_n), \ldots</td>
<td>1, 2, 3, \ldots, n, \ldots</td>
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### Pfaffian Transform Examples

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**Conjecture**

If \{a_j\} (eventually) satisfies a linear homogeneous recurrence relation with constants coefficients, then so does \text{Pf}(\{a_j\}).
Theorem

If $A$ is skew-symmetric and we obtain $B$ from $A$ by
1. adding a multiple of row $i$ to row $j$ and
2. adding the same multiple of column $i$ to column $j$
then $B$ is skew-symmetric and $\text{Pf}(A) = \text{Pf}(B)$. 
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\[
\text{Pf}(\{ F_j \})_3 = \text{Pf} \begin{pmatrix}
0 & 1 & 1 & 2 & 3 & 5 \\
-1 & 0 & 1 & 1 & 2 & 3 \\
-1 & -1 & 0 & 1 & 1 & 2 \\
-2 & -1 & -1 & 0 & 1 & 1 \\
-3 & -2 & -1 & -1 & 0 & 1 \\
-5 & -3 & -2 & -1 & -1 & 0 \\
\end{pmatrix}
\]
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\[
\text{Pf}(\{F_j\})_3 = \text{Pf}
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0 & 1 & 1 & 2 & 3 & 0 \\
-1 & 0 & 1 & 1 & 2 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 \\
-2 & -1 & -1 & 0 & 1 & 0 \\
-3 & -2 & -1 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & -2 \\
\end{pmatrix}
\]
Theorem

If $A$ is skew-symmetric and we obtain $B$ from $A$ by

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$$\text{Pf}(\{F_j\}_3) = \text{Pf} \begin{pmatrix} 0 & 1 & 1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ -2 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}$$
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\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 \\
\end{pmatrix}
\]

\[
\text{Pf}(\{ F_j \})_3 = \text{Pf}
\]
A Reduction with Row and Column Operations

**Theorem**

If $A$ is skew-symmetric and we obtain $B$ from $A$ by

1. adding a multiple of row $i$ to row $j$ and
2. adding the same multiple of column $i$ to column $j$

then $B$ is skew-symmetric and $\text{Pf}(A) = \text{Pf}(B)$.

**Conclusion:** We can assume $a_j = 0$ for all large $j$. 
The Claw Graph
The Claw Graph
The Claw Graph
The Claw Graph

![Diagram of the Claw Graph with nodes and edges labeled]
The Claw Graph
The Claw Graph
The Claw Graph
The Claw Graph

\begin{enumerate}
  \item [1]  
  \item [2]  
  \item [3]  
  \item [4]  
  \item [5]  
  \item [6]  
  \item [7]  
  \item [8]  
  \item [9]  
  \item [10]  
\end{enumerate}
The Claw Graph
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The Pfaffian Transform

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The Claw Graph
The State Digraph

Definition

The $k$-claw on $2n$ vertices is the graph with vertices $1, 2, \ldots, 2n$ in which vertices $i$ and $j$ are adjacent whenever $0 < |i - j| \leq k$. 
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Idea

Terms in $\text{Pf}(x_1, \ldots, x_k, 0, \ldots)_n$ are indexed by perfect matchings in the $k$-claw on $2n$ vertices.
The State Digraph

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The $k$-claw on $2n$ vertices is the graph with vertices $1, 2, \ldots, 2n$ in which vertices $i$ and $j$ are adjacent whenever $0 < |i - j| \leq k$.

Idea

Terms in $\text{Pf}(x_1, \ldots, x_k, 0, \ldots)_n$ are indexed by perfect matchings in the $k$-claw on $2n$ vertices.

Idea

These perfect matchings are in bijection with paths in a certain digraph, called the state digraph.
States in the 3-Claw
States in the 4-Claw
The Adjacency Matrix for the 3-Claw

The Adjacency Matrix for the 3-Claw is given by

\[ A_3 = \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2 & -x_3 & 0 & 0 \\ x_1 & 0 & 0 & -x_3 \\ x_3 & 0 & 0 & 0 \end{pmatrix} \]

Eric S. Egge (Carleton College)
Deus Ex Transfer Matrix Method

**Theorem**

\[
\sum_{n=1}^{\infty} \text{Pf}(x_1, x_2, \ldots, x_k, 0, \ldots)_n t^n = \frac{\det(l - tA_k; 1, 1)}{\det(l - tA_k)}
\]
Deus Ex Transfer Matrix Method

Theorem

\[ \sum_{n=1}^{\infty} \text{Pf}(x_1, x_2, \ldots, x_k, 0, \ldots) n t^n = \frac{\det(l - tA_k; 1, 1)}{\det(l - tA_k)} \]

Corollary

If \{a_j\} satisfies a linear recurrence relation with constant coefficients then so does \text{Pf}(\{a_j\}).
Deus Ex Transfer Matrix Method

**Theorem**

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**Corollary**

*If \( \{a_j\} \) satisfies a linear recurrence relation with constant coefficients then so does \( \text{Pf}(\{a_j\}) \).*

**Bonus:** We can find the recurrence relation in terms of \( x_1, \ldots, x_k \).
The End

Thank You!