Teaching general relativity to undergraduates

Nelson Christensen and Thomas Moore

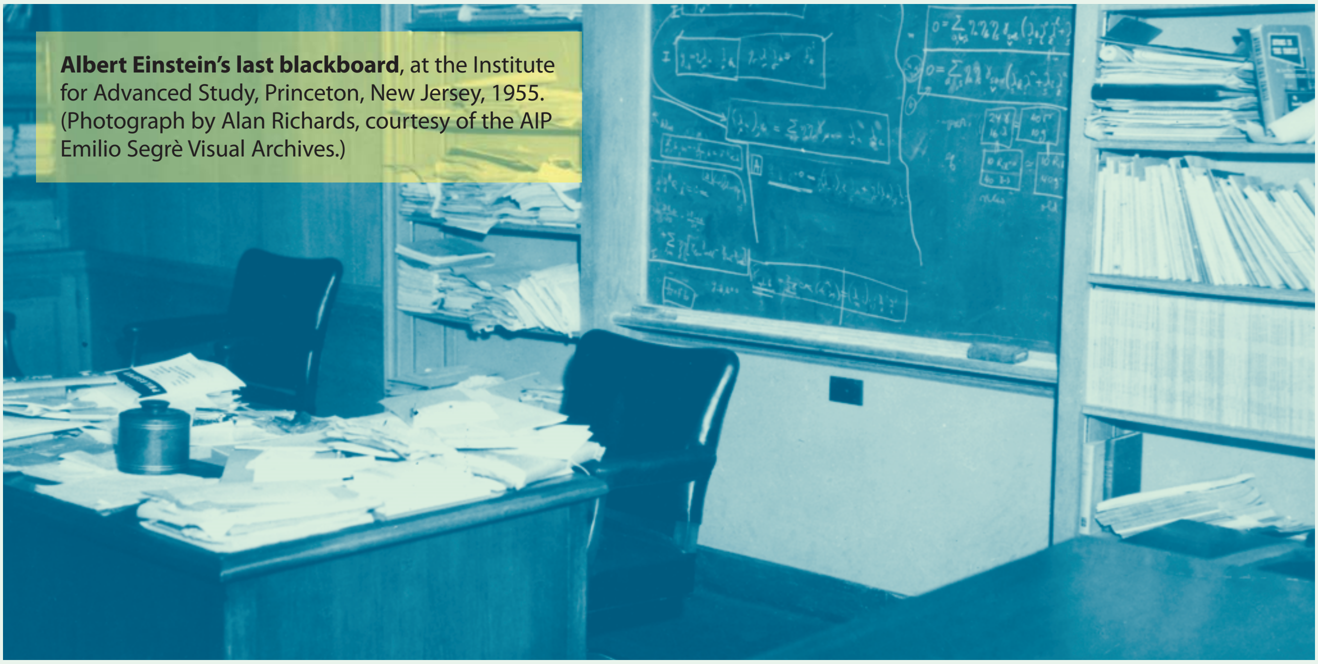
Inspired by new results in cosmology and astrophysics, undergraduates are increasingly eager to learn about general relativity. A number of innovative textbooks make it easier than ever before to satisfy that demand.

General relativity lies at the heart of a wide variety of exciting astrophysical and cosmological discoveries made during the past decade or so. Whereas in 1919 Arthur Eddington was allegedly at a loss to name a third person who understood general relativity, presently its knowledge and use are widespread. We have come a long way in a century.

Perhaps the most compelling of the new results is the 1998 discovery that not only is the universe expanding, but the expansion is accelerating. The 2011 Nobel Prize in Physics was awarded for observations of high-redshift supernovae whose distance and brightness revealed the accelerated expansion and therefore showed that the cosmos is filled with "dark energy" (see PHYSICS TODAY, December 2011, page 14). Observations of the cosmic microwave background have also provided evidence for dark energy—and for nonluminous dark matter. Because the revolution in the scientific community's understanding of the universe has been widely reported, undergraduate students are interested in learning more about cosmology.

New astrophysical observations are also piquing students' curiosity. Astronomers observe supermassive black holes in the centers of galaxies, including our own Milky Way. We now recognize that neutron stars are abundant in our galaxy and that they provide the basis for extreme and intriguing astrophysical phenomena. The Laser Interferometer Gravitational-Wave Observatory and the Virgo interferometer are expected to see gravitational waves within a few years. That discovery will be an important confirmation of general relativity in its own right; moreover, the gravitational waves will provide a new means to view the universe and will reveal relativistic events in regions shielded from electromagnetic observation. General relativity is also important close to home. Gravity Probe B, which had been orbiting Earth, recently observed frame-dragging and geodetic effects predicted by general relativity. The global positioning system, which students regularly use, requires general relativity to ensure its meter-scale accuracy (see the article by Neil Ashby in PHYSICS TODAY, May 2002, page 41). In a way that was not true even two decades ago, general relativity has become an issue of practical concern to mainstream physicists and even engineers.[1]

Nelson Christensen (nchriste@carleton.edu) is a professor of physics at Carleton College in Northfield, Minnesota. **Thomas Moore** (tmoore@pomona.edu) is a professor of physics at Pomona College in Claremont, California.



Albert Einstein's last blackboard, at the Institute for Advanced Study, Princeton, New Jersey, 1955. (Photograph by Alan Richards, courtesy of the AIP Emilio Segrè Visual Archives.)

The popular press has reported on all of the above. Not surprisingly, undergraduate students are taking notice and wanting to better understand the physics. They want to learn general relativity to engage the science both in the classroom and through research projects.

Increasingly, college and university teachers are working to create appropriate courses in general relativity for undergraduate physics majors, aided by a number of textbooks that offer new strategies for successfully introducing the subject at a reasonable pace and level. Indeed, our experience is that such a course need not be limited to the most gifted students; undergraduates at the level of junior or senior physics majors are generally quite capable of learning general relativity in satisfying depth. In fact, we encourage all institutions offering undergraduate physics degrees to seriously consider providing a semester-length general relativity course for their students. And to help make that happen, we will share several pedagogical strategies and describe our and others' experiences of student success with those strategies.

Teaching approaches

Historically, the reason general relativity has not been taught to undergraduates is that the subject has been considered prohibitively difficult. The full theory of general relativity is based on the concepts of differential geometry, most often expressed in the language of tensor calculus; it thus involves mathematics beyond what most undergraduates encounter.

We have identified four distinct approaches that textbook authors have used to address that difficulty. We call them

- The adjusted math-first approach.
- The calculus-only approach.
- The physics-first approach.
- The intertwined + active-learning approach.

The adjusted math-first approach was the first to be developed. It adopts the same basic outline as a graduate-level course, including an early-on full treatment of tensor calculus and the mathematics of curved spacetime. But it adjusts the presentation of that mathematics to be more appropriate for mature undergraduates. Excellent texts of that type include *A First Course in General Relativity* by Bernard Schutz and *Gravitation and Spacetime* by Hans Ohanian and Remo Ruffini. (For bibliographic information for these and all general relativity textbooks cited in this article, see the box on page 44.)

Almost all undergraduate general relativity texts published in the 21st century turn the focus decisively away from the math and toward the physics. But even within that common philosophy, the mathematical challenge remains, and it is handled differently in each of the remaining three approaches.

The calculus-only approach follows an innovative trajectory developed in the final decades of the 20th century and described by Edwin Taylor and John Wheeler in their book *Exploring Black Holes*. In the calculus-only approach, spacetime metrics are given, not derived, and the focus is on extracting the implications of the geodesic equation for those metrics. Students who have completed an introductory calculus-based physics course can explore many of the most interesting physical consequences of general relativity without tensors or even multivariable calculus.

The physics-first approach is perhaps best exemplified by James Hartle's innovative 2003 text, *Gravity*. Like the calculus-only approach, it focuses on working through the implications of given metrics, but at the mathematical level of most junior and senior undergraduates. (It does include some gently developed tensor calculus.) Hartle's extraordinary breadth of knowledge concerning applications of general relativity helps to make the physical examples in his text extraordinarily rich and varied. Although Hartle's book strongly emphasizes the physics, its final chapters provide the full mathematics required to understand the Einstein equation; however, Hartle has designed his book so that those mathematical sections can be omitted.

The last of the approaches, intertwined + active-learning, is based on the experimentally supported hypothesis that junior and senior undergraduates can indeed learn the tensor mathematics needed to fully understand general relativity—if the

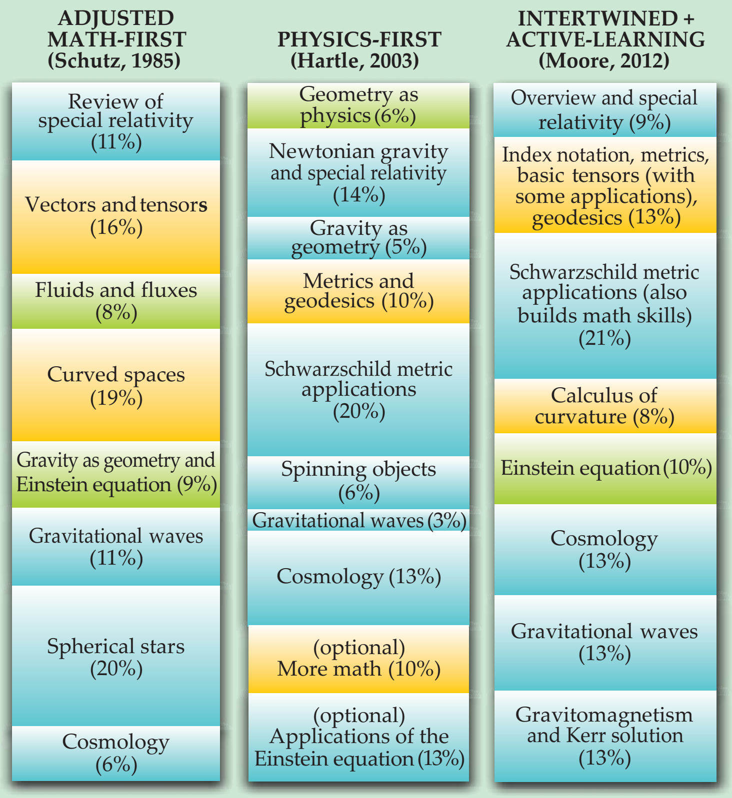


Figure 1. The relative emphasis of physics and mathematics and their positioning in exemplary texts for the adjusted math-first, physics-first, and intertwined + active-learning approaches. All three texts are designed for a one-semester, upper-level course. Blue sections are primarily physics oriented, yellow sections are primarily math oriented, and green sections blend some of each. Cited percentages are based on page counts.

instructor develops the math slowly, on an as-needed basis thoroughly intertwined with the physics; presents that math at a level really suited to undergraduates; and uses active-learning techniques to ensure that the students get the individual practice needed to own both the math and the physics. As in the calculus-only and physics-first approaches, the physics is strongly emphasized over the math. The approach is exemplified by the soon to be published *A General Relativity Workbook* by one of us (Moore).

Our four categories are admittedly a bit fuzzy and may not always capture the features of individual textbooks. Still, we think that they highlight important differences worth understanding. Figures 1 and 2 illustrate some of those differences. The first summarizes and contrasts the exemplary texts for the three approaches that target upper-level undergraduates. Because the calculus-only approach is qualitatively different, we have not included Taylor and Wheeler's text. The second figure displays a number of books, including some calculus-only and graduate texts, in terms of their degree of mathematical sophistication and willingness to delay presenting the physics.

When we listed the four approaches, we presented them in approximate chronological order of development. However, even though later books react to what has gone before, one should not take "later" to be the same as "better." Rather, the approaches each widen the potential audience; together they support courses for undergraduates with a broad range of interests and abilities. Many of our colleagues have reported satisfaction and success with whichever approach they chose.

Adjusted math-first

Traditionally, graduate instruction in general relativity has followed a math-first approach. In the 1970s and 1980s, Ohanian and Schutz offered the first serious attempts to follow that traditional approach to target undergraduates. The material in their books mostly parallels the classic graduate-level texts of the era, but the authors treat that material with substantially less rigor without, in Schutz's words, "watering down the subject matter." With the mathematics established, the field equations can be derived and then solved for cases of astrophysical interest. The application of the theory to physical systems comes at the end. Ohanian and Ruffini have developed an interesting variant on the adjusted math-first approach: They initially concentrate on a linearized approximation to general relativity, study some applications in low-curvature situations, and then move on to the full general-relativistic theory. Their variant allows for an early discussion of gravitational waves and light deflection.

We found relatively few undergraduate courses in the US that currently use the adjusted math-first approach, and a substantial fraction of those are taught in mathematics departments, in which emphasizing the underlying mathematical concepts before turning to physics makes good sense. The approach is also used in physics courses designed for both senior undergraduates and beginning graduate students.

Calculus-only

The textbook that first developed, and most vividly exemplifies, the calculus-only approach is Taylor and Wheeler's *Exploring Black Holes*. Figure 3 gives an overview of its basic approach and topics; Taylor's own description of its goal is more succinct: "Undergraduate general relativity developed near the black hole using calculus, no tensors." The book makes no attempt to develop the full mathematical machinery of general relativity or the Einstein equation. Rather, it explores given metrics, teaching students how to extract meaning from them and to predict the motion of particles in their spacetimes. The focus, instead of being on equations, is very much on developing a student's conceptual understanding.

Even so, Taylor and Wheeler, through clever and judicious simplifications, manage to calculate geodesics with only simple calculus and the principle of maximal aging—the postulate that a free particle follows the world line of maximum proper time. Although Taylor and Wheeler give metrics without derivation, they work out the physical implications of those metrics carefully and completely. Their approach is similar to and roughly at the same level as the treatment of the Schrödinger equation in a typical modern physics course in which students do not see the equation's deep roots as a position-basis expression of the system's Hamiltonian operator but do get some motivation as to why it might make sense and then work out implications first in simple one-dimensional situations and then in successively more complicated cases.

Taylor and Wheeler's book, like their special-relativity classic *Spacetime Physics*,[2] provides

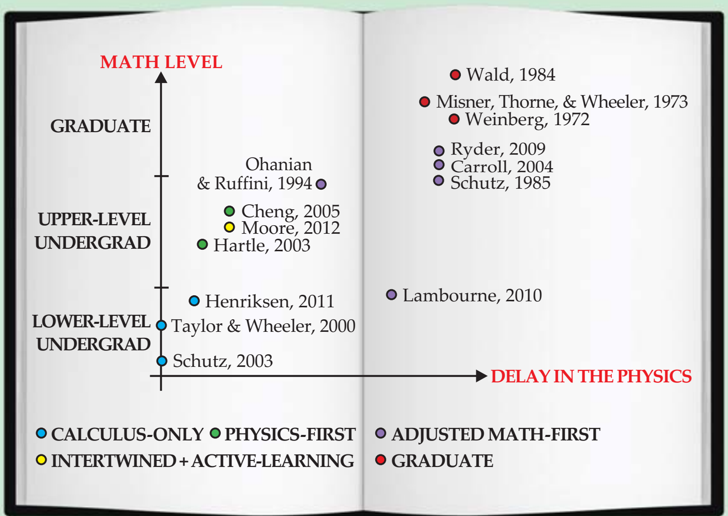


Figure 2. The main sequence. General relativity textbooks differ with regard to their level of mathematical sophistication and how long it takes for them to get to physical applications beyond special relativity. This is a limited sampling, and locations are somewhat subjective; still, they may provide for useful comparisons.

Some recommended relativity texts

Many texts can form the basis for an excellent undergraduate course in relativity. Here are some that we recommend, along with a smattering of classic graduate-level books.

- S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York (1972).
- C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).
- R. M. Wald, *General Relativity*, U. Chicago Press, Chicago (1984).
- B. F. Schutz, *A First Course in General Relativity*, Cambridge U. Press, New York (1985; 2nd ed. 2009).
- H. C. Ohanian, R. Ruffini, *Gravitation and Spacetime*, 2nd ed. Norton, New York (1994). The first edition was written by Ohanian alone in 1976.
- E. F. Taylor, J. A. Wheeler, *Exploring Black Holes: Introduction to General Relativity*, Addison Wesley Longman, San Francisco (2000).
- J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Addison-Wesley, San Francisco (2003).
- B. F. Schutz, *Gravity from the Ground Up*, Cambridge U. Press, New York (2003).
- S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, San Francisco (2004).
- T.-P. Cheng, *Relativity, Gravitation and Cosmology: A Basic Introduction*, Oxford U. Press, New York (2005, 2nd ed. 2010).
- L. Ryder, *Introduction to General Relativity*, Cambridge U. Press, New York (2009).
- R. J. A. Lambourne, *Relativity, Gravitation and Cosmology*, Cambridge U. Press, New York (2010).
- R. N. Henriksen, *Practical Relativity: From First Principles to the Theory of Gravity*, Wiley, Chichester, UK (2011).
- T. A. Moore, *A General Relativity Workbook*, University Science Books, Sausalito, CA (in press). See http://pages.pomona.edu/~tmoore/grw.

amazing physical insights at an introductory level. It is especially good about making a sharp distinction between the arbitrary nature of global coordinates and things that a local observer can actually measure. Most other books do not draw that distinction nearly so well.

We corresponded with a number of faculty members who used *Exploring Black Holes* in courses for undergraduates who had only a basic background in introductory physics and calculus. Some of those courses used *Spacetime Physics* as well, to provide students with a solid introduction to both special and general relativity. Several instructors took advantage of the projects that the book suggests to give students a chance to explore something on their own. All reported how exhilarated students were to be seriously exploring the implications of relativity. In various ways, our correspondents stated that the book provides an excellent opportunity for students with a minimal physics and math background to discover a lot of interesting physics and that it gives students strong motivation for further study.

Several later books have, in quite different ways, targeted audiences with calculus or even precalculus math skills. One of the most ambitious and insightful is Schutz's *Gravity from the Ground Up*, which is suitable for students interested in self study or for a course geared toward students not majoring in physics or math.

Physics-first

It was Hartle who coined the term "physics first," to describe his own approach, summarized in figure 1, to teaching general relativity to undergraduates.[3] Thanks in large part to his work, the physics education community has devoted much attention to that approach. A physics-first class would usually be at the junior- or senior-year level and would require higher math skills than needed for a calculus-only course. As in a typical upper-division electricity and magnetism course, students in a physics-first course use the additional math to delve deeper into the subject.

Nonetheless, the course emphasizes concepts, and a goal is to not go too deep into the tensor calculus—at least, not right away—but rather to get to the interesting physics quickly. A geometric presentation of special relativity given early in the course highlights the metric and invariant quantities and motivates the general relativistic physics. The text introduces important metrics, rather than deriving them, and then uses those metrics for an in-depth exploration of such important physical phenomena as particle orbits and the deflection of light rays. The results from those calculations are compared with experimental results and astrophysical observations; indeed, the connection to current experiments and observations provides strong encouragement for the students. The final part of the course can be dedicated to motivating the Einstein equation and solving it for the spacetime geometries previously discussed.

We have found that the majority of undergraduate general relativity classes taught in physics departments use the physics-first approach. The teachers for those classes at a wide variety of institutions report very good outcomes and positive feedback from students. Some of the students in a physics-first class will be energized to pursue general relativity further, so teachers using that approach should be prepared to provide further references.

Intertwined + active-learning

There are good reasons to give students a full introduction to tensor calculus. In our experience, students often feel that they are on somewhat shaky ground in a course that focuses on results such as metrics, gravitational-wave formulas, and gravitomagnetic phenomena but that does not provide the foundation and context for those results. But designing a course that presents the full tensor calculus is no easy task. Making undergraduates comfortable with the math takes time and effort, but delaying the gratification of interesting physics for many weeks would sap the students' and instructor's motivation. In addition, students need time for the nonintuitive concepts of general relativity and the dizzying new tensor notation to sink in. Interweaving the math and physics throughout the course is one way to meet the challenge. That approach is exemplified by Moore's *A General Relativity Workbook*, summarized in figure 1.

Even when appropriately spread out, the mathematics presents real difficulties for undergraduates. The intertwined + active-learning approach addresses that particular challenge by pushing students to work out the math (and physics) themselves, so that they come to own the math in a way that would not otherwise be possible. Each chapter in *A General Relativity Workbook* is meant to correspond to a single class day and typically consists of four pages of text that provide an accessible conceptual overview of the day's material without getting sidetracked by derivations or other detailed arguments. The students themselves, guided by cues, work out all derivations and details in a series of boxes. Teachers can thus devote class time to addressing their students' specific difficulties.

Both of us have used the intertwined + active-learning approach, and we usually begin a class session by asking students which box exercises were most challenging. We then invite selected students to the board to present their work on those challenging exercises. Discussions typically follow, in which students exchange ideas and techniques and we offer guidance. That format allows us to give students feedback and help exactly where it is needed. Often we have time to work example homework problems or discuss the physical and mathematical issues raised in the chapter.

We have found that active learning enables students to progress much farther and become much more confident than do the more passive approaches we have used. In fact, we regard active learning as a crucial part of the intertwined approach and a technique by which students in the same target audience as the physics-first approach can enjoy the benefits of increased mathematical sophistication. But it is only suitable for instructors comfortable with active-learning methods.

Tools for success

Between the two of us, we have taught undergraduate general relativity courses about 20 times and have used all four of the approaches discussed above. We have learned a great deal about what works and what does not. Here we share some ideas and resources for you to keep in mind when developing an undergraduate general relativity course based on any of the approaches.

Choosing appropriate prerequisites is important. Most professors using the calculus-only approach require the equivalent of one year of introductory physics and one year of introductory calculus. The other approaches require more background. In our experience, and in that of most of our correspondents, vector calculus is an absolute minimum. In addition, to develop the appropriate level of physics sophistication, students will need a class that goes beyond the typical sophomore modern-physics course.

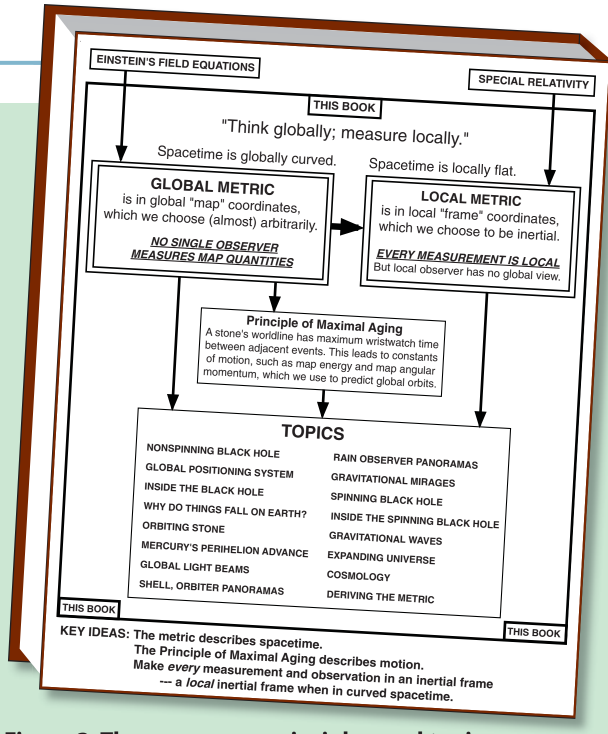


Figure 3. The scope, core principles, and topics covered in Edwin Taylor and John Wheeler's calculus-only text, *Exploring Black Holes*. This summary is the anticipated back cover of the forthcoming second edition. (Courtesy of Edwin Taylor.)

Undergraduates gain valuable experience from laboratory-based classes in which they learn to make measurements and tie the results to underlying physics. When teaching general relativity, we always have the students ask themselves what they can measure. Class examples and homework problems should cause students to think constantly about taking measurements with meter sticks and clocks in a stationary or freely falling laboratory or in a distant observatory. Like Albert Einstein himself, your students will have trouble understanding that the spacetime coordinates involved in global coordinate systems have no meaning other than what the metric gives them. Therefore, as emphasized by Taylor and Wheeler (see figure 3), it is important to help students draw the distinction between global coordinates and real physical measurements performed in a local laboratory. Fortunately, that emphasis is more or less built into the calculus-only, physics-first, and intertwined + active-learning approaches.

The emphasis on measurement need not be limited to theory. Your undergraduate general relativity course will almost certainly not have an accompanying laboratory class. Nonetheless, you can connect some well-known undergraduate experiments with the course and provide opportunities for demonstrations or projects in which students actually take data. Measuring the speed of light is always instructive. Determining the muon lifetime illustrates how subatomic particles can be used as clocks that measure proper time. Many smartphones have apps that display readings from the device's three-axis accelerometer; throwing the smartphone onto a soft cushion and then analyzing the logged data can help clarify the concept of a freely falling reference

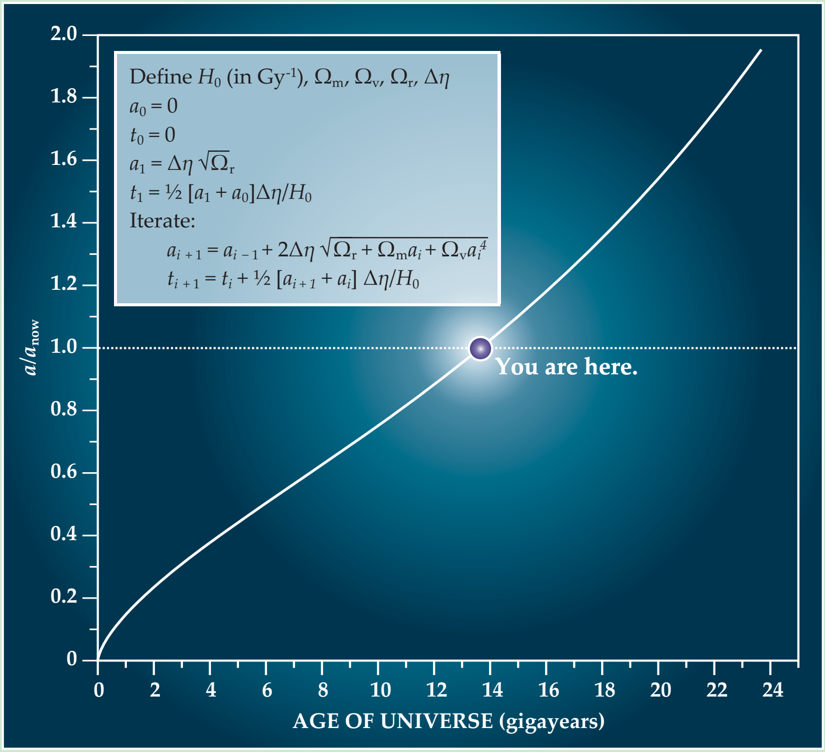


Figure 4. How to construct a universe. Even with a simple spreadsheet, students can construct an accurate numerical model for the expansion of our universe, as described by the scale parameter *a*. This graph was generated from spreadsheet data assuming a flat universe whose current Hubble constant is given by $1/H_0 = 13.7$ Gy. The current cosmic energy fractions in matter (Ω_m), vacuum energy (Ω_v), and radiation (Ω_r) are 0.272, 0.728, and 0.000084, respectively. The algorithm is based on a pair of simple difference equations representing the differential equations for the scale factor and time *t* as a function of the parameter η. Note the inflection point at about 9 Gy. That is the time at which vacuum energy begins to dominate matter and the cosmic expansion accelerates.

frame.[4] Students can use a relatively inexpensive radio telescope to actually measure the galactic rotation curve and so discover that our galaxy contains dark matter.[5] Although we are unaware of relativity experiments that use the global positioning system, the many relativistic effects underlying GPS make for interesting study and discussion.[1]

Computer advances have made it possible for undergraduates to do calculations and explore physical problems that would have been much more daunting even a decade ago. Hartle has posted some Mathematica-based programs on the webpage for his textbook.[6] With them a student can easily generate Christoffel symbols and curvature tensors for any desired metric. Free computer programs on various sites let students calculate orbits in the Schwarzschild and Kerr geometries, plot expansion rates for the universe for different cosmological models,[7,8] or explore how physicists estimate the amount of dark matter in a galaxy.[9]

Several books, including Schutz's *Gravity from the Ground Up* and Moore's *Workbook*, explicitly discuss how to construct simple numerical models. Figure 4 shows the output for a spreadsheet-based model of cosmic expansion that includes realistic amounts of vacuum energy (the cosmological-constant form of dark energy), matter, and radiation. We have found such assignments extremely valuable in helping students make predictions about our universe, as distinguished from the toy models usually explored in textbooks. Just as important, such exercises help students learn about the process of constructing numerical models. Moreover, by wrestling with such models, students understand the physics better: For one thing, they have to learn how to recognize when an incorrect model is producing unphysical results.

Don't spare the drill

The mathematics involved in general relativity is not that much different from what students have encountered in their studies of vectors. In a sense, tensors are just big vectors, and we have found it valuable to make as many connections as possible to vector calculus. But the notation truly does look different, and in any approach other than calculus-only, students need to become comfortable with that difference. Spend some time asking the students to identify free and summed indices, write out the implied sums in an equation, check that the free indices on both sides of an equation are consistent, rename indices in an appropriate way to pull out a common factor, and identify nonsensical equations. It may seem silly to drill students on such basic things. Our experience, however, is that spending time with such simple drills and addressing standard errors when first teaching the notation really do improve student competence and confidence. Few textbooks, in our opinion, provide sufficient attention to such drills.

We have also found 2D visualizations to be helpful. Students should practice applying every new tensor concept to easily visualized 2D flat and curved spaces before using it in a 4D spacetime. They can work with polar coordinates in flat space and longitude–latitude coordinates on a curved sphere, then move on to stranger coordinate systems for flat space and other types of curved spaces.

Practicing the formalism and applying it to concrete, easily visualizable systems helps students to avoid errors and build an intuition about and confidence in the mathematics.

Whatever approach you favor, you should try to use active-learning methods in your class. One of the most robust results of physics education research is that students learn the material much more effectively when they are actively engaged in the learning process instead of passively receiving the material in lectures or readings.[10] Students need to take ownership of ideas by actively processing them and holding them up against their own experience. Most of the work demonstrating the efficacy of active learning has focused on introductory courses, but we have found that our upper-level students exposed to active-learning techniques perform much better than those in more traditional courses. Fortunately, most general relativity classes are likely to be small enough that active learning should be reasonably easy to implement. Still, to do that, you will need to make sure that students come to class prepared, design appropriate activities for both in and out of class, and give students appropriate feedback on their work.

Advances in astronomy and cosmology have opened a whole new world of opportunities that is waiting for undergraduate physics students with a background in general relativity. Those who teach the subject can take advantage of a decade's worth of pedagogical progress. Various educational techniques long used in teaching electrodynamics and quantum mechanics to undergraduates can now be successfully applied to general relativity, and new textbooks are making such approaches broadly accessible. General relativity is therefore rapidly becoming as important as many topics traditionally covered for a physics degree. We strongly recommend that any major-granting department consider a regular course offering in this fascinating subject.

We thank the numerous physics teachers and textbook authors who have helped us acquire information for this article.

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